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A RESUME OF DETERMINISTIC TIME-VARYING LINEAR SYSTEM THEORY WITH APPLICATION TO ACTIVE SONAR SIGNAL PROCESSING PROBLEMS

by

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15 DECEMBER 1980

NORTH ATLANTIC TREATY ORGANIZATION

LA SPEZIA, ITALY

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(6) A RESUME OF DETERMINISTIC TIME-VARYING LINEAR SYSTEM THEORY WITH APPLICATION TO ACTIVE SONAR SIGNAL PROCESSING PROBLEMS,

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10) Lewis Meier

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A RESUME OF DETERMINISTC TIME-VARYING LINEAR SYSTEM THEORY WITH APPLICATION TO ACTIVE SONAR SIGNAL PROCESSING PROBLEMS

by

Lewis Meier

ABSTRACT

Brief summaries are given of time-varying linear system theory in both the convention frequency-shift version and the novel frequency-shift version. The latter version of the theory is utilized to obtain the matched filter output for the bistatic echo from a moving, turning line target for a linear FM pulse in a non-dispersive medium.

INTRODUCTION

The major concern of this report is with a sonar return processed in the standard manner with a beam former (not considered herein) and a bank of matched filters, where the modification of the transmitted signal by the medium and the target can be represented by a time-varying linear system. Two transmitted signals are of interest: the CW pulse, which has unity time/bandwidth product and the FM pulse, which has a time/bandwidth product much greater than unity. Correspondingly this report summarizes two versions of deterministic time-varying linear system theory: the standard frequency-shift version, applicable to analysis of detection and measurement extraction using CW pulses and the more novel frequency stretch (doppler) version, applicable to analysis of detection and measurement extraction using FM pulses.

The approach is to characterize the various parts of the signal path between a sonar transmitter and a sonar receiver as linear time-varying systems. The theory then provides means to characterize the signal at various point along the path, to characterize the parts of the path between these points and to relate these characterizations. Specifically the signals are characterized by cross-ambiguity functions (called the ambiguity function for the transmitted signal), which tell what the response to the signals of a bank of matched filters would be as a function of time shift and either frequency shift or frequency stretch; the parts of the path are characterized by spreading functions, which tell how the parts spread the signal in time shift and frequency shift or frequency stretch and the relationship is a modified double convolution in time shift and frequency shift or frequency stretch.

Rihaczek *[1]* made an extensive study of the use of matched filter radars and since he was primarily interested in point targets and a perfect medium, he investigated in great detail ambiguity functions of the transmitted signal. In fact his book could be titled "All You Wanted to Know about Ambiguity Functions but were Afraid to Ask". The use of the standard theory was applied to sonar problems by Søstrand *[2]*, whose development is followed here with minor modifications. Kelley and Wishner *[3]* studied generalization of the classical ambiguity function including the doppler ambiguity function studied herein. They were also interest in the situation in which acceleration of the target caused a significant change in velocity during the duration of the transmitted pulse. While acceleration can be very important with targets such as missiles, it seldom is so in the sonar situation and, hence, is neglected herein. The complete development of the frequency-stretch theory presented here is believed to be novel.

Weston *[4]*, Kramer *[5]* and Russo and Bartberger *[6]* have all derived the doppler ambiguity function of a linear FM pulse in terms of Fresnel Integrals with differing limits of integration. Russo and Bartberger give the exact limits, which are rather complicated, while Weston uses a gross

approximation to obtain the general behaviour of the doppler ambiguity function not only for linear FM pulses, but curved FM pulses as well. Weston's paper is highly recommended for its physical insight into the nature of ambiguity functions of FM pulses. Kramer (and T. Kooij in unpublished work of 1964) uses limits of integration that are valid approximation for time shifts small compared to the pulse lengths; herein the next level of approximation to the limits, which allows large time shifts but ignores changes in length of the signal, is derived. This approximation is valid for normal sonar situations when the time-bandwidth product of the FM signal is less than 750. Furthermore it is shown that when in addition the time-bandwidth product exheeds 50, the Fourier transform with respect to time shift of the doppler ambiguity function of a linear FM pulse takes a particularly simple form.

A good and useful example of the application of the time-shift theory presented herein is determination of the matched filter output for the bistatic echo from a moving, turning line target for a linear FM pulse, when all medium effects other than propagation delays are ignored. The doppler spreading function of such a target may be readily approximated in a simple form and then used to obtain the matched filter output as a convolution between an impulse response function representing the target and a window function representing the FM pulse. This result, derived herein, is also believed to be novel and provides the basis in a sequel for algorithms for measuring target properties.

Two topics I will not be concerned with in this report are propagation loss and background. The sonar equation and its extension may be utilized to determine level of the echo with respect to background at the matched-filter-bank input; I am concerned with how that echo is spread in time and frequency or Doppler at the matched-filter-bank output. Both pieces of information are required for investigating detection and measurement extraction. The theory presented herein and in a sequel on stochastic systems have direct application to the analysis of reverberation, but I will not pursue that topic either herein or in the sequel.

The main results are contained in the body of the report with the detailed derivations relegated to appendices. The body of the report is divided into two sections: the first is concerned with development of the theory while the second is concerned with its application to finding the matched filter output of the bistatic echo from a moving turning target for a linear FM transmission in a non-dispersive medium.

1 THE THEORY

Figure 1 is a simplified block diagram of a single beam of an active sonar system. The matched filters bank consists of a set of filters matched to the transmitted signal $x(t)$ shifted in frequency by different amounts. In practice, for a CW pulse the matched filter bank is realized by taking a time sequence of Fourier transforms of segments of echo whose length approximately equals that of the pulse, while for an FM pulse only a single matched filter matched to the transmitted signal is used because of the large doppler tolerance of this signal. This form of processing is predicated on the assumption that the doppler effect is a frequency shift, when in actual fact it is a frequency stretch; therefore, the matched filter bank should consist of a set of filters matched to $x(t)$ stretched in frequency by different amounts. Fortunately for an CW pulse the frequency stretch is adequately approximated by a frequency shift while for an FM pulse use may be made of the equivalence between zero frequency shift and unity frequency stretch.

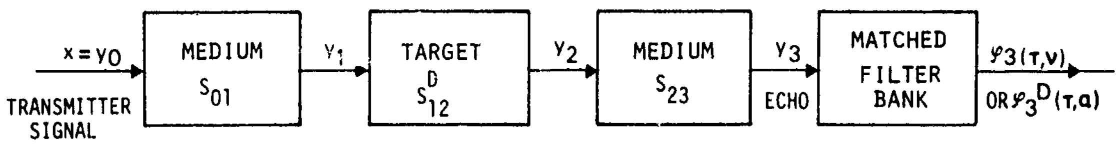

FIG. 1 SIMPLIFIED BLOCK DIAGRAM OF A SINGLE BEAM OF AN ACTIVE SONAR SYSTEM

1.1 The cross-ambiguity function

Since we are trying to analyze the matched-filter-bank ouput, it is reasonable to characterize the signal $y(t)$ at various points between transmission and reception by what output it would yield if used as input for the matched-filter bank. This function of the time shift τ and the frequency shift ν or the frequency stretch α is formally known as the cross-ambiguity function ϕ_i or ϕ_i^D. Throughout the report the superscript D — for doppler — will be used to denote the frequency stretch version of variables. In Fig. 1, the cross-ambiguity function ϕ_3 or ϕ_3^D of the echo is the matched-filter bank output, while the ambiguity function of the input $\gamma = \phi_o$ or $\gamma^D = \phi_o^D$ (where the cross prefix is dropped for obvious reasons) tells how the transmitted signal is spread in τ and ν or α. The cross-ambiguity functions ϕ_i and ϕ_i^D are defined by

$$\phi_i(\tau,y) = \int_{-\infty}^{\infty} e^{-2\pi j\nu(t-\tau)}\; x^*(t-\tau)\; y_i(t)\, dt \quad , \qquad \text{(Eq. 1a)}$$

$$\phi_i^D(\tau,\alpha) = \alpha \int_{0}^{\infty} x^*[\alpha(t-\tau)]\; y_i(t)\, dt. \qquad \text{(Eq. 1b)}$$

where * denotes complex conjugate.

1.2 The spreading function

The effect of the target on the signal is to shift it in time and doppler due to its range and range rate and to spread it in time and doppler due to its apparent length and apparent turning rate. It is of course these effects that can be used to measure these properties of a target as Plaisant has so ably investigated *[7,8]*. On the other hand, the effect of the medium on the signal is to spread it in time and frequency due to multipaths and motion in the medium. Since a time-varying linear system such as a target or the medium modifies its input $y_I(t)$ in time and frequency or doppler; it is not unreasonable to suppose that its output $y_0(t)$ be a weighted sum of the input shifted in time and either shifted or stretched in frequency:

$$y_0(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j\nu(t-\tau)} y_I(t-\tau)\, S_{0I}(\tau,\nu)\, d\nu d\tau \;, \qquad \text{(Eq. 2a)}$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} y_I[\alpha(t-\tau)]\, S^D_{I0}(\tau,\alpha)\, d\nu d\tau \;. \qquad \text{(Eq. 2b)}$$

Equations 2a and 2b define the spreading functions $S_{0I}(\tau,\nu)$ and $S^D_{0I}(\tau,\alpha)$ of the system producing $y_0(t)$ from $y_I(t)$.

It is more standard to describe a time-varying linear system by its impulse response $h_{I0}(t,\tau)$, which is defined as the system output at time t due to an impulsive input at time $t-\tau$. Use of an impulsive input $y_I = \delta(t-\tau)$ in Eqs. 2a or 2b yields:

$$h_{I0}(t,\tau) = \int_{-\infty}^{\infty} e^{2\pi j\nu(t-\tau)}\; S_{I0}(\tau,\nu)\; d\nu \;, \qquad \text{(Eq. 3a)}$$

$$= \int_{0}^{\infty} \frac{1}{\alpha}\; S^D_{I0}\left(\frac{(\alpha-1)\,t+\tau}{\alpha}, \alpha\right) d\alpha \;. \qquad \text{(Eq. 3b)}$$

From the similarity of Eq. 3a to a Fourier transform it is clear that it can be inverted to yield

$$S_{I0}(\tau,\nu) = \int_{-\infty}^{\infty} e^{-2\pi j\nu(t-\tau)}\; h_{I0}(t,\tau)\; dt. \qquad \text{(Eq. 4)}$$

By its very linearity every time-varying linear system has an impulse response; therefore it has also an ordinary spreading function. On the other hand, it is apparent from the impossibility of inverting (Eq. 3b) that not every time-varying linear system has a Doppler spreading function. Nonetheless some very important systems do have Doppler spreading functions.

1.3 The fundamental relationship

Of key importance to the analysis is the relationship between the cross-ambiguity functions ϕ_I or ϕ_I^D and ϕ_O or ϕ_O^D of the input and output of a linear time-varying system as characterized by its spreading function S_{IO} or S_{IO}^D ; this relationship is a modified convolution in τ and ν or α as illustrated in Fig. 2. By applying this result recursively starting with ϕ_o or ϕ_o^D (the ambiguity function of the transmitted signal), ϕ_1 or ϕ_1^D , ϕ_2 or ϕ_2^D and finally ϕ_3 or ϕ_3^D (the cross-ambiguity function of the echo) may be computed. Alternatively the relationship between the overall spreading function S_{AG} or S_{AC}^D of the two systems with spreading functions S_{AB} or S_{AB}^D and S_{BC} or S_{BC}^D, which is the same modified convolution in τ and ν or α as is illustrated in Fig. 3, may be applied recursively to find the spreading function between transmitted signal and echo and then the original relationship between transmitted signal and echo used to find ϕ_3 or ϕ_3^D.

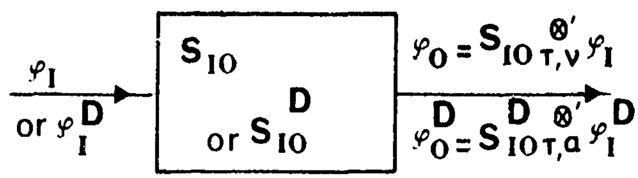

FIG. 2 RELATIONSHIP BETWEEN THE INPUT AND INPUT CROSS-AMBIGUITY FUNCTIONS AND THE SPREADING FUNCTION OF A TIME-VARYING LINEAR SYSTEM

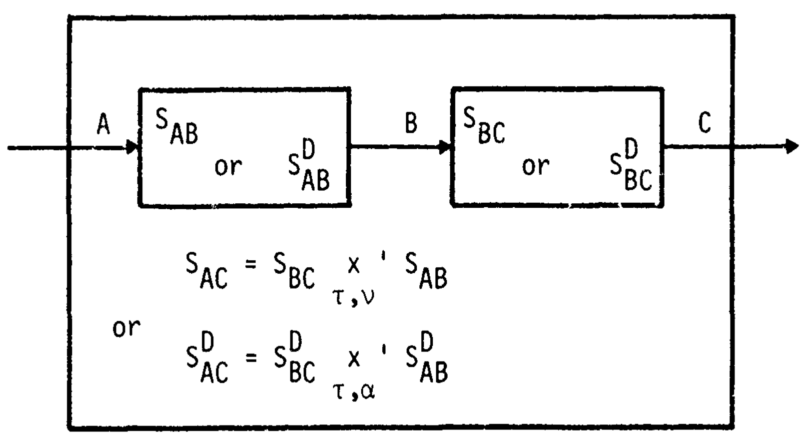

FIG. 3 THE SPREADING FUNCTION OT TWO SYSTEMS IN TAMDEM IN TERMS OF THEIR INDIVIDUAL SPREADING FUNCTIONS

The modified double convolution operators $\underset{\tau,\nu}{\otimes}'$ and $\underset{\tau,\alpha}{\otimes}'$ required are defined by

$$b \underset{\tau,\nu}{\otimes}' a(\tau,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j\nu'(\tau-\tau')} a(\tau-\tau',\nu-\nu') \; b(\tau',\nu') \, d\nu' \, d\tau' , \quad \text{(Eq. 5a)}$$

$$b^D \underset{\tau,\alpha}{\otimes}' a^D(\tau,\alpha) = \int_{-\infty}^{\infty}\int_{0}^{\infty} a^D[\alpha'(\tau-\tau'), \tfrac{\alpha}{\alpha'}] \; b^D(\tau',\alpha') \, d\alpha' \, d\tau' . \quad \text{(Eq. 5b)}$$

By rewritting the definitions (Eqs. 1a, 1b, 2a and 2b) in terms of these modified convolutions, it may be shown that the relationships depicted in Figs. 2 and 3 hold.

1.4 Some approximate relationships

As will be seen below the ordinary ambiguity function of a signal such as an FM or CW pulse is much simpler in form than its doppler ambiguity function while, on the other hand, the doppler spreading function of a target is much simpler than the ordinary spreading function. Let B be the bandwidth, T the duration and f_0 the carrier of the signal.

For sufficiently small signal time bandwidth product BT and target time stretch α sufficiently near 1, namely for

$$BT \; |\alpha - 1| < 0.2 \quad \text{(Eq. 6)}$$

the frequency stretch α may be approximated by a frequency shift $(\alpha-1)f_0$ since in this case the maximum phase error caused by so doing is negligible.

When Eq. 6 holds, it follows that

$$S_{IO}(\tau,\nu) \approx \frac{1}{f_0} S^D_{IO}(\tau, 1 + \frac{\nu}{f_0}) . \quad \text{(Eq. 7)}$$

Note that Eq. 7 is exact for $\alpha = 1$ (i.e. $\nu = 0$), since the approximation is exact for $\alpha = 1$.

Clearly Eq. 6 usually holds for a CW signal (with $BT = 1$) in typical sonar and radar applications. On the other hand, for the linear FM in typical sonar applications as opposed to typical radar applications for which $|\alpha-1|$ is usually much smaller —(6) often does not hold.

From the foregoing it is obvious that for a low BT signal, such as a CW pulse, analysis of sonar detection and measurement extraction may be carried out using the ordinary theory with the exact spreading function replaced by an approximate spreading function found from the Doppler spreading function of the target. On the other hand for a high BT signal such as a linear FM pulse the exact spreading function, which is apt to be much more complicated, must be used if the ordinary theory is applied. One can regain a simple form for the spreading function, at the cost of a moderate increase in complexity of the ambiguity function, by using the frequency stretch version of the theory.

There is, however, one fly in the ointment -- it is more convenient to represent the time and frequency spreading effects of the medium via the ordinary spreading function. Fortunately, it is possible to approximate $\phi_I^D(\tau,\alpha + \frac{\nu}{f_o})$ for $\nu < \frac{0.2\ f_o}{BT}$ as an ordinary cross-ambiguity function and thus show that

$$\phi_O^D(\tau,\alpha) \approx e^{2\pi j\nu'(\tau-\tau')} \frac{1}{\alpha - \frac{\nu}{f_o}} \phi_I^D(\tau-\tau',\alpha- \frac{\nu'}{f_o}) S_{IO}(\tau',\nu')\ d\nu'\ dt' \qquad \text{(Eq. 8)}$$

if $S_{IO}(\tau',\nu')$ is negligible for $\nu' > 0.2\ f_o/BT$. For example, if $f_o = 3{,}500$ Hz, $B = 220$ Hz and $T = 0.5$s; this limit is roughly 7 Hz (or a Doppler shift of ± 3 knots), which should be sufficient to include most frequency spreading in the ocean. Use of this approximation allows us to use the Doppler-shift version for large BT signals while using the ordinary spreading function to model the medium.

2 SOME APPLICATIONS OF THE THEORY

In this section some applications of the theory just summarized are given both to illustrate the theory and because of their own inherent interest. The goal of these applications is to derive the relationship between the matched filter output for the bistactic echo from a linear FM pulse for a moving, turning line target in a non-dispersive medium. To apply the fundamental relationship to determine the matched filter output, it is necessary first to use the definitions to obtain the ambiguity function of a linear FM pulse and the bistatic spreading function of a moving, turning line target. Along the way it is convenient to include the propagation delays in the target spreading function.

2.1 The ambiguity functions of a linear FM pulse

As an example of an ambiguity function consider the modulated linear FM pulse

$$x_{FM}(t;f_0,k,T) = \frac{1}{\sqrt{T}} \sqcap (t;T/2)\, e^{2\pi j(f_0 t + \frac{1}{2}kt^2)} , \qquad \text{(Eq. 9)}$$

where f_0 is the carrier frequency, T is the pulse length, $B = |k|T$ is the bandwidth and

$$\sqcap(t;T) = \begin{cases} 1 & \text{if } |t|<T \\ 0 & \text{if } |t|>T \end{cases} . \qquad \text{(Eq. 10)}$$

Its ambiguity functions are

$$\gamma_{FM}(\tau,\nu;\ f_0,k,T) = \left(1 - \frac{|\tau|}{T}\right)\sqcap(\tau,T)\, e^{2\pi j(f_0+\frac{1}{2}\nu)\tau}\ \mathrm{sinc}[\pi(T-|\tau|)(k\tau-\nu)]$$

$$\approx e^{2\pi j(f_0+\frac{1}{2}\nu)\tau}\ \mathrm{sinc}[\pi\, T(k\tau-\nu)] \quad \text{for} \quad \tau \ll T .$$

(Eq. 11a)

$$\gamma^{D}_{FM}(\tau,\alpha;\ f_0,k,T) = e^{2\pi j(\frac{\alpha+1}{2})f_0\tau}\ \gamma^{M}_{FM}\left[\alpha\tau-(\alpha-1)\frac{f_0}{k} , \frac{1}{\alpha} ;\ f_0,k,T\right] \qquad \text{(Eq. 11b)}$$

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where an approximation to γ_{FM}^{M} will be given shortly. The ordinary ambiguity γ_{FM} given in Eq. 11a is well known and needs no comment beyond the fact that the ordinary ambiguity function of a CW pulse is obtained by setting $k = 0$ in Eq. 11a. The doppler ambiguity function γ_{FM}^{D} is expressed in Eq. 11b in terms of γ_{FM}^{M} because the matched filter output for a linear FM signal frequency stretched by α_0, from Fig. 2 and Eq. 5b (since S_{IO}^{D} in this case is $\delta(\tau)\ \delta(\alpha-\alpha_0)$), is

$$\gamma_{FM}^{D}(\alpha_0\tau, \frac{1}{\alpha_0}) = e^{2\pi j(\frac{\alpha_0+1}{2})f_0\tau}\ \gamma_{FM}^{M}\left(\tau + \frac{\alpha_0-1}{\alpha_0}\frac{f_0}{k}, \alpha_0\right) . \qquad \text{(Eq. 12)}$$

Thus if $(\alpha_0+1)\ f_0/2$ is taken as the carrier of the matched filter output γ_{FM}^{M} represents the modulation; it is time shifted so that its peak in τ for given α is near $\tau = 0$.

When γ_{FM}^{M} is found by substitution of x_{FM} for both x and y in Eq. 1b and use of the result in Eq. 12, it is can be expressed in terms of Fresnel integrals with rather complicated limits; however suitable approximation yields simpler limits that are valid for most sonar purposes. It is convenient to give the results in terms of the following normalized variables:

$$\begin{aligned} x &\triangleq kT\sqrt{T/k_0}/\alpha \approx kT\sqrt{T/f_0} , \\ y &\triangleq (\alpha-1)\sqrt{f_0T/\alpha} \approx (\alpha-1)\sqrt{f_0T} , \\ \nu &\triangleq \alpha\sqrt{1/f_0T} \approx \sqrt{1/f_0T} , \\ z &\triangleq k\,T\,\tau \end{aligned} \qquad \text{(Eq. 13)}$$

In general f_0 and T (and hence ν) are determined by external circumstances and the choice of k is available for optimizing the properties of γ_{FM}^{M}; therefore x, y and z are referred to as the normalized bandwidth (since $|kT| = B$), normalized time stretch and normalized time shift respectively.

In terms of these variables

$$\gamma_{FM}^{M}(z,y;\ x,\nu) = \tfrac{1}{2}\sqcap\left[\left(y - \frac{\nu z}{1+\nu y}\right)/x, 1+\nu y\right]\frac{e^{-\frac{\pi}{2}j\phi}}{(1+\nu y)\sqrt{xy}}$$

$$\cdot\{c(b)-c(a)+j\ \mathrm{sgn}(xy)[S(b)-S(a)\} + \varepsilon \qquad \text{(Eq. 14)}$$

where

$$C(t) + j\,S(t) = \int_{-\infty}^{t} e^{\frac{\pi}{2}jt^2}\,dt \quad ,$$

$$a = \sqrt{|xy|}\left[\frac{z}{xy} - (1+\nu y) + \left|(y - \frac{\nu z}{1+\nu y})/x\right|\right] \quad ,$$

$$b = \sqrt{|xy|}\left[\frac{z}{xy} + (1+\nu y) - \left|(y - \frac{\nu z}{1+\nu y})/x\right|\right] \quad ,$$

$$\phi = \frac{z^2}{xy} - xy\left[(y - \frac{\nu z}{1+\nu y})/x\right]^2 \; ,$$

$$\varepsilon < |y|\,\nu\left(\frac{1}{1+\nu y} + 0.27|xy|\right)$$

and sgn( ) means sign of. Not that for $\nu = 0$ these relationships are both much simpler and exact (i.e. $\varepsilon = 0$).

It will be seen that the maximum value $|\alpha-1|$ of interest is 0.02. In terms of the original variables $\varepsilon < |\alpha-1|(\frac{1}{\alpha} + 0.27\ BT\ |\alpha-1|)$; therefore for $BT < 750$, ε can be neglected with respect to the maximum of γ^M_{FM}, which is from Eqs. 1a, 1b, 11a and 11b $\gamma^M_{FM}(0,0;\ x,\nu) = 1$.

Since $\gamma^D_{FM}(\alpha\tau,\ 1/\alpha)$ is the output of filter matched to x_{FM} when its input is x_{FM} frequency stretched by α , its fourier transform with request to τ is the product of the matched filter transfer function times the fourier transform of the frequency stretched signal. For large BT — which for fixed x corresponds to small ν — these both have the same form: quadratic in phase and rectangular in magnitude; hence the fourier transform of $\gamma^D_{FM}(\alpha\tau,\ 1/\alpha)$ with respect to τ is readily obtained. This result may then be time and frequency shifted in accordance with Eq. 12 to yield $\Gamma^M_{FM}(f,\alpha)$ the fourier transform of $\gamma^M_{FM}(\tau,\alpha)$ with respect to τ. (If the fourier transform is made with respect to the normalized variable z instead of τ the result is $B\,\Gamma^M_{FM}$; it is convenient to ignore this factor and consider Γ^M_{FM} to be the fourier transform of γ^M_{FM} with respect to z expressed as a function of the normalized frequency $w = f/kT$.) The result in normalized variables

$$\Gamma^M_{FM}(w,y;\ x,\nu) \approx \sqcap(y/x;\ 1)\,\sqcap(2w;\ 1-|\tfrac{y}{x}|)\ e^{2\pi j(xy\,w^2 + y^3/4x)} \qquad \text{(Eq. 15)}$$

This expression is exact for $\nu = 0$ and, in fact its inverse fourier with respect to w is Eq. 14 with . = 0 — as it must be.

To determine how strong the dependence of γ_{FM}^{M} and Γ_{FM}^{M} are on ν, $\max_z |\gamma_{FM}^{M}(z_o y ; x,\nu) - \gamma_{FM}^{M}(z,y ; x,o)|$ was computed for various x y and ν. The results are shown in Fig. 4, which gives this maximum error as a function of $y\,\nu/0.02 = |\alpha-1|/0.02$ for various x and x/ν = BT. The maximum value $|\alpha-1|$ used was 0.02, since, as mentioned earlier, this will be seen to be the maximum value of interest. As might be expected the error is strongly dependent only upon BT.

From Fig. 4 it is clear that for BT > 50 the error in using $\nu = 0.0$ is negligible compared to the peak value of γ_{FM}^{M}, which is $\gamma_{FM}^{M}(0, 0 ; x,\nu) = 1$. One should not expect, however, that for such values of BT, Γ_{FM}^{M} will take the nice form given in Eq. 15. In fact because of Gibb's like phenomenon the actual magnitude and phase will oscillate around the rectangular magnitude and quadratic phase of Eq. 15. Nevertheless it is reasonable to say that $\Gamma_{FM}^{M}(w,y ; x,o)$ approximates $\Gamma_{FM}^{M}(w,y,x,\nu)$ whenever $\gamma_{FM}^{M}(z,y ; x,o)$ approximates $\gamma_{FM}^{M}(z,y ; x,\nu)$.

The quadratic phase dependence in Eq. 15 implies that the frequencies in γ_{FM}^{M} differing from the carrier are shifted in time with respect to the carrier; i.e. that the peak of γ_{FM}^{M} for a given y is broadened in y. This effect is shown graphically in Figs. 5, 6 and 7, which are graphs of $|\gamma_{FM}^{M}|^2$ versus z for $\nu = 0$ and various x and y. In fact the fourier transform of $|\gamma_{FM}^{M}(z,y ; x,o)|^2$ with respect to y is readily found by autocorrelation of $\Gamma_{FM}^{M}(w,y ; x,o)$ with respect to w to be

$$\tfrac{1}{2} \sqcap (y/x\ ;1) \sqcap (w,\ 1-|y/x|)(1-|y/x|-|w|)\mathrm{sinc}[2\pi x\ y\ w(1-|y/x|-|w|)] \quad .$$

(Eq. 16)

Two limiting cases are of interest: for small xy the last factor is essentially unity for $|w|<1-|x,y|$ and $|\gamma_{FM}^{M}|^2$ is just the inverse fourier transform of the first two factors with respect to w ; namely $(1-|y/x|)^2\ \mathrm{sinc}^2[\pi(1-|y/x|)z]$ — which is of course the classical result found when the doppler effect may be approximated by a frequency shift. On the other hand for large xy the first two factors in Eq. 16 may be replaced by unity and the last approximated by $\sin\ [2\pi|xy|w(1-|x/y|)]$;

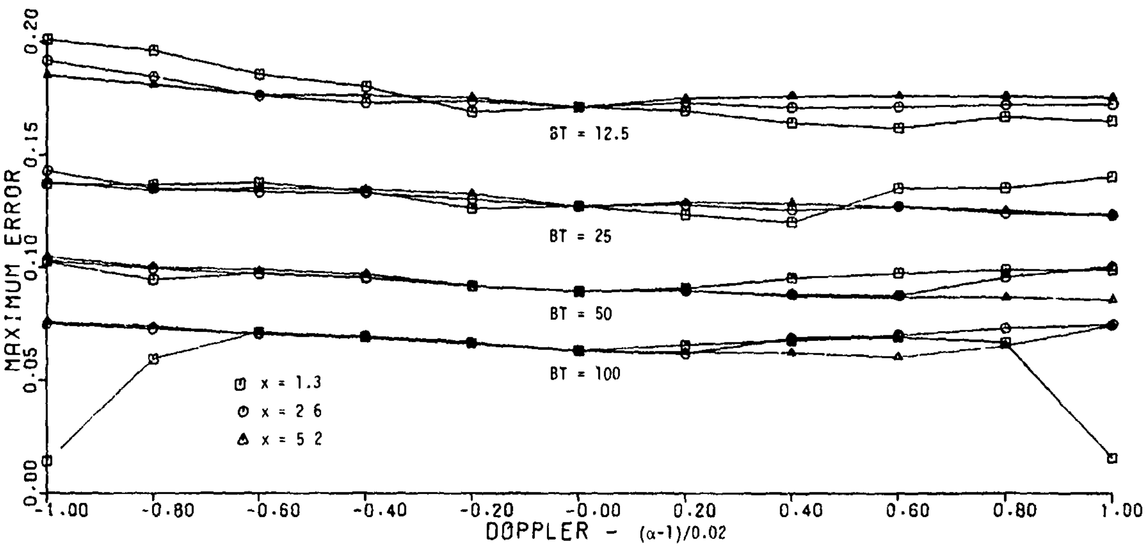

FIG. 4 $\max_z |\gamma^M_{FM}(z,y;\ x,v) - \gamma^M_{FM}(z,y;\ x,o)|$ vs $y/0.02$ $v = (\alpha-1)/0.02$

for various x/v = BT and x

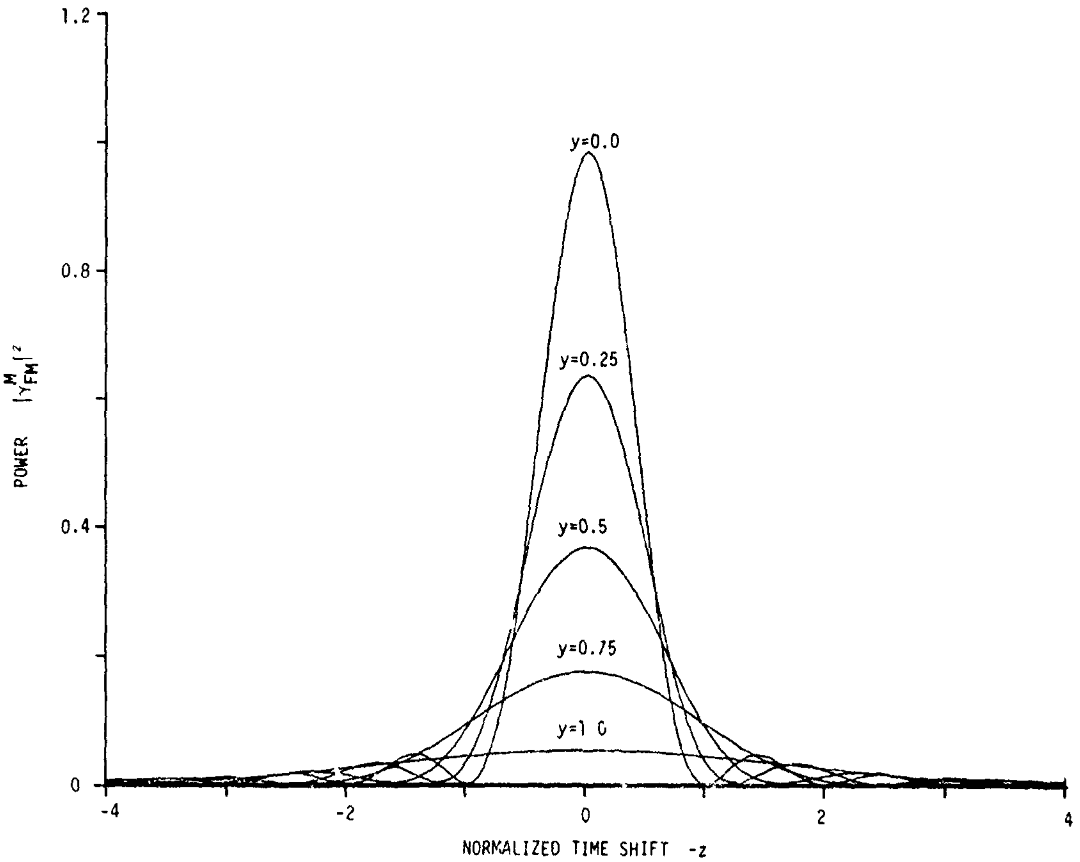

FIG. 5 $|\gamma^M_{FM}|^2$ VS Z FOR X = 1.3 AND VARIOUS Y

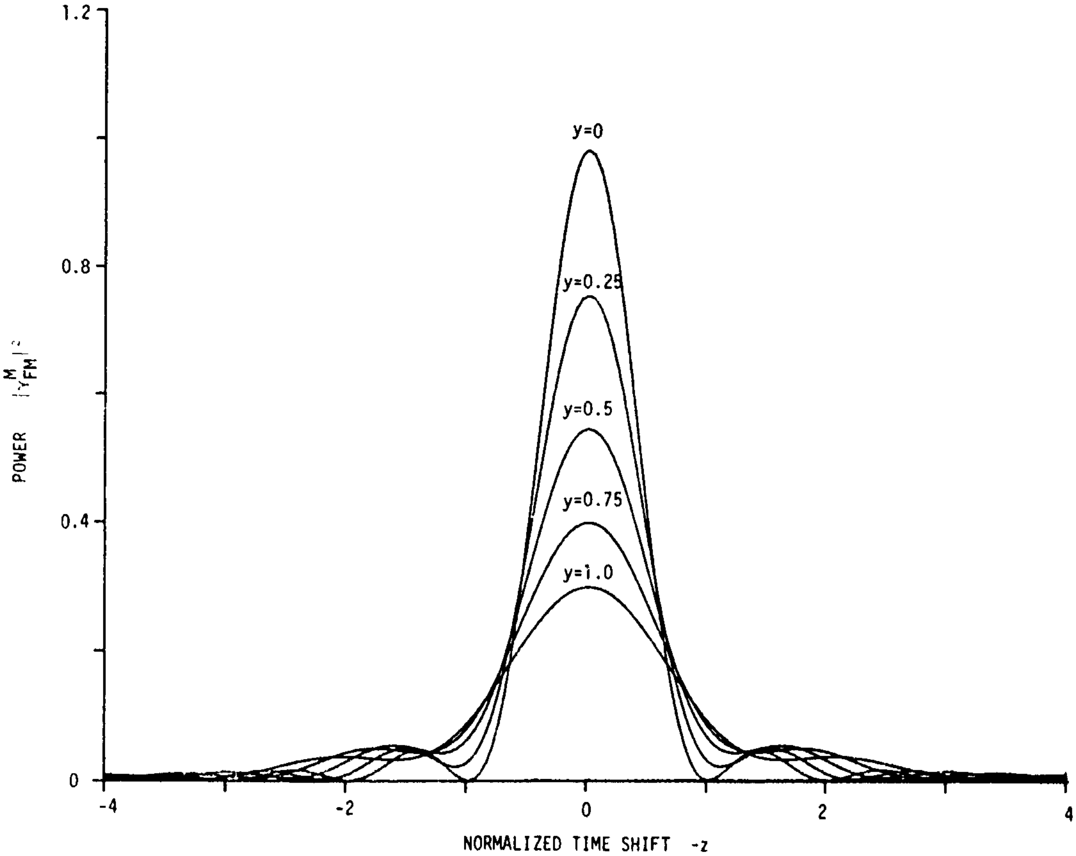

FIG. 6 $|\gamma^M_{FM}|^2$ VS Z FOR X = 2.6 AND VARIOUS Y

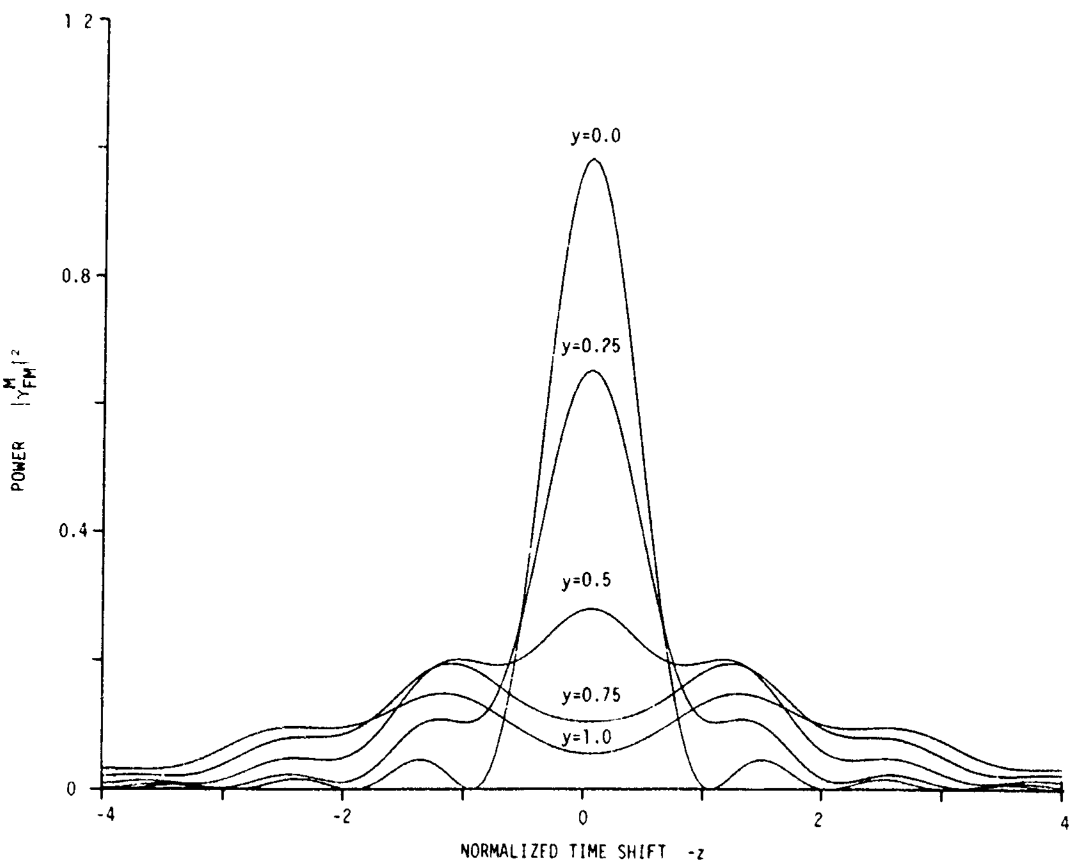

FIG. 7 $|\gamma^M_{FM}|^2$ VS Z FOR X = 5.2 AND VARIOUS Y

hence $|\gamma_{FM}^{M}|^2$ is $\dfrac{1}{|yx|(1-|y/x|)} \sqcap [z \; ; \tfrac{1}{2}|yx|(1-|y/x|)]$ — which is the result obtained by Weston *[4]*.

The FM doppler ambiguity function just discussed in detail is for a pulse whose center corresponds to $t = 0$. In section 2.4 the FM doppler ambiguity function $\gamma_{FM}^{D'}(\alpha\tau, 1/\alpha)$ for a pulse whose center has an abitrary time t_T is required. Furthermore, in that section a simple relationship between the doppler ambiguity function for α close to α_o and that for α_o is required. If the definition of the doppler ambiguity function (Eq. 1b) is applied for an FM pulse centered on t_T and the frequency stretch α replaced by a frequency stretch α_o plus a frequency shift $(\alpha-\alpha_o)f_o$, the result after suitable algebric manipulation and approximation is

$$\gamma_{FM}^{D'}(\alpha\tau, 1/\alpha) = e^{-2\pi j\left(\frac{\alpha-\alpha_o}{\alpha_o}\right)\frac{f_o^2}{k}} \gamma_{FM}^{D}\left[\alpha_o\left(\tau + \frac{\alpha-\alpha_o}{\alpha_o^2\, k} f_o + \frac{\alpha-1}{\alpha} t_T\right) , 1/\alpha_o\right] , \qquad \text{(Eq. 17)}$$

which is valid if

$$|\alpha-\alpha_o| < \min \; \frac{0.2}{BT} , \; 0.05\,\frac{B}{f_o} , \; \frac{0.2B}{f_o}\sqrt{\frac{1}{BT}} \; . \qquad \text{(Eq. 18)}$$

2.2 Factoring propagation delays into the target spreading function

An important application of the fundamental relationships illustrated in Fig. 3 is to make a convenient modification to the medium and target spreading functions. In their most straightforward versions the propagation delays to and from the target are included in the medium spreading functions. It is more convenient to transfer the direct path delays into the target spreading function so that with a non-dispersive medium only the target spreading function need be used to relate the matched-filter-bank outputs to the ambiguity function of the transmitted signal. In this version the medium spreading functions are only used to represent dispersive effects of the medium.

The procedure to be carried out is illustrated in Fig. 8 for the frequency shift version where $S_{01'}$, $S_{1'2'}$ and $S_{2'3}$ are the spreading functions for

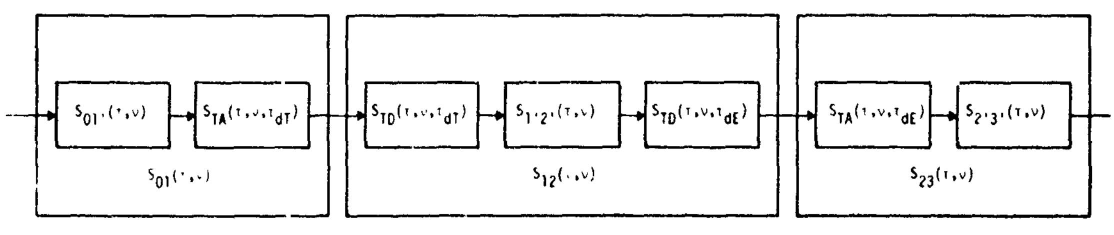

FIG. 8 RELATIONSHIPS BETWEEN TWO VERSIONS OF A SIMPLIFIED ACTIVE SONAR MODEL

the straightforward version (delays in the medium spreading functions); S_{01}, S_{12} and S_{23} are the spreading functions for the more convenient version (delays in the target spreading function) and $S_{TD}(\tau,\nu;\,\tau_d)$ and $S_{TA}(\tau,\nu;\,\tau_d)$ are the spreading functions of a time delay and a time advance by τ_d. Obviously S_{TA} followed by S_{TD} or vice versa cancel each other. The results from use of Fig. 3 and Eq. 5a to combine spreading functions are as follows:

$$S_{01}(\tau,\nu) = S_{0'1'}(\tau+\tau_{dT},\nu) \quad ,$$

$$S_{12}(\tau,\nu) = e^{-2\pi j\tau dT}\, S_{1'2'}(\tau-\tau_{dT}-\tau_{dE},\nu) \quad ,$$

$$S_{23}(\tau,\nu) = e^{2\pi j\tau dE}\, S_{2'3}(\tau+\tau_{dE},\nu) \quad , \qquad \text{(Eq. 19a)}$$

where τ_{dT} and τ_{dE} are the transmission and echo direct path propagation delays. From Fig. 8 it is obvious that $S_{12}(\tau,\nu)$ is the spreading function of the overall system for a non-dispersive medium.

For the time stretch version the same procedure may be used to yield

$$S^D_{01}(\tau,\alpha) = S^D_{0'1'}(\tau+\tau_{dT},\alpha) \quad ,$$

$$S^D_{12}(\tau,\alpha) = S^D_{1'2'}(\tau-\frac{\tau_{DT}}{\alpha} - \tau_{DE},\alpha) \quad ,$$

$$S^D_{23}(\tau,\alpha) = S^D_{2'3}(\tau+\frac{\tau_{dE}}{\alpha},\alpha) \quad , \qquad \text{(Eq. 19b)}$$

as before, $S^D_{12}(\tau,\alpha)$ is the Doppler spreading function of the overall system for a non-dispersive medium.

2.3 The bistatic spreading function of a moving, turning line target

The target model and geometry to be investigated is shown in Fig. 9, where the variables used are defined in Table 1. The values of the variables of interest are those that obtain at a given time that may conveniently be taken to be $t = 0$. It is assumed that the medium is noise and reverberation free, has no propagation loss and does not spread the signal in time or frequency. Taking into account noise, reverberation and propagation loss presents no particular problem and may be done essentially independently

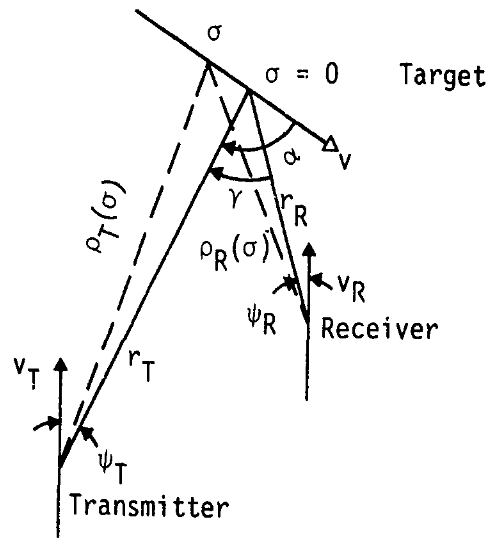

FIG. 9 BISTATIC GEOMETRY

TABLE 1

LIST OF VARIABLES

α	Target aspect
γ	Bistatic angle
ψ_T	Bearing of target with respect to transmitter
ψ_R	Bearing of target with respect to receiver
v	Target speed
v_T	Transmitter speed
v_R	Receiver speed
r_T	Range from transmitter to centre of target
r_R	Range from receiver to centre of target
σ	Distance along target
$\rho_T(\sigma)$	Range from transmitter to point at σ along target
$\rho_R(\sigma)$	Range from receiver to point σ along target

of the development presented herein; however including time and frequency spreading requires a modification of the present development using the techniques described above. One point of possible confusion is the use of α for both the aspect and the doppler shift; fortunately the context makes clear which use is implied.

For short periods of time each point along the target may be assumed to have constant range rates with respect to the transmitter and receiver that are readily determined from the geometry. The corresponding time stretches and hence the doppler spreading functions for each point may then be determined and summed to yield the target spreading function. The result under the assumation that the signal arrives at the target near $t = 0$ is for a target whose points have reflectivity $f(\sigma)$

$$S^D(\tau,\alpha) \approx \frac{c}{(1-a)\delta}\, f\left(\frac{-c\tau+\ell-\frac{\alpha-1}{\alpha}r_T}{(1-a)\ \delta}\right)\delta\left[\alpha-1+\frac{\dot{\ell}}{c} - \frac{\beta}{1-\alpha}\left(\tau-\frac{\ell}{c}+\frac{\alpha-1}{\alpha}\,\frac{r_T}{c}\right)\right] \tag{Eq. 20}$$

where $\delta(\ )$ is the Dirac delta function, c is the speed of sound and

$$\ell \triangleq r_T + r_R \quad ,$$

$$\dot{\ell} \triangleq \dot{r}_T + \dot{r}_R = -(v\delta + v_T \cos\psi_T + v_R\,\psi_R) \quad ,$$

$$\delta \triangleq [\cos\alpha + \cos(\alpha-\gamma)] \quad ,$$

$$a \triangleq \dot{\ell}\cos\alpha/c\delta \quad ,$$

$$\beta \triangleq [\dot{\alpha}\sin\alpha + (\dot{\alpha}-\dot{\gamma})\sin(\alpha-\gamma)]/\delta \quad . \tag{Eq. 21}$$

For a monostatic situation the same results apply with $\gamma \equiv 0$, $r_T = r_R = r$, $\psi_T = \psi_R = \psi$ and $v_T = v_R = v_{TR}$; hence

$$\ell = 2r \quad ,$$

$$\dot{\ell} = 2\dot{r} = -2(v\cos\alpha + v_{TR}\cos\psi) \quad ,$$

$$\delta = 2\cos\alpha \quad ,$$

$$a = \dot{\ell}/2c \quad ,$$

$$\beta = \dot{\alpha}\tan\alpha \quad . \tag{Eq. 22}$$

2.4 The matched filter output for the bistatic echo from a linear FM pulse for a moving, turning line target in a non-dispersive medium

To find the match filter output $\phi_3^D(\tau,1)$ it is necessary to substitute the approximate FM ambiguity and target spreading functions given in Eqs. 17 and 20 into the fundamental relationship defined by Fig. 2 and Eq. 5b. Recall that t_T is the center of the FM pulse. If $t_T = -r_T/c$ the signal will arrive at the centre of the target at $t = 0$; therefore let $\Delta t_T = t_T + r_T/c$ so that $\Delta t_T = 0$ implies $t_T = -r_T/c$. The results to be given only apply for small Δt_T say on the order of the length of a single ping (which may include several pulses)* since only then will (Eq. 20) be valid. The result of the substitutions with $\alpha_o = 1 - \ell/c$ is

$$\phi_3^D(\tau,1) \approx e^{2\pi j\, f_o'(\tau-\tau_d')}\, \psi^M(\tau-\tau_d) \qquad \text{(Eq. 23)}$$

where

$$f_o' = (1 - \frac{\dot{\ell}}{2c})\, f_o \;,$$

$$\tau_d' = (\ell + \frac{c\dot{\ell}}{c-\dot{\ell}}\, \Delta t_T)/c \;,$$

$$\tau_d = [\ell + \frac{c\dot{\ell}}{c-\dot{\ell}}(\Delta t_T + \frac{f_o}{k})]/c \;,$$

$$\Psi^M(f) = \mathcal{F}\,[\psi^M(\tau)] = \Gamma_{FM}^M(f,\, 1-\dot{\ell}/c)H^M(f) \;,$$

$$\Gamma_{FM}^M(f,\alpha) = \mathcal{F}\,[\gamma_{FM}^M(\tau,\alpha)] \;,$$

$$H^M(f) = F(S\{[1 - \frac{c^2\beta}{(c-\dot{\ell})^2}\, \Delta t_T - \alpha](f+f_o') - \frac{c^2\beta}{(c-\dot{\ell})^2}\,\frac{f_o}{k}\,(f + \frac{\dot{\ell}}{2c}\, f_o)\}/c) \;,$$

$$F(f) = \mathcal{F}\,[f(-\tau)] \;, \qquad \text{(Eq. 24)}$$

and $\mathcal{F}$ [] indicates Fourier transform of .

**In general t_h may not be known. This is not important so long as the signal and matched filter are timed to the same reference.*

Severall points may be observed about this hodgepodge:

- The carrier of the matched filter output is both time and frequency shifted as a function of $\dot{\ell}$

- The modulation of the matched filter output is shifted in time by a different amount as a function of $\dot{\ell}$

- Transformed into the frequency domain the modulation of the matched filter output is the frequency shifted and stretched target transfer function $F(f)$ observed through the window $\Gamma^M_{FM}(f)$, which has been discribed above.

- Since $[c/(c+\dot{\ell})]^2 \approx 1$, the effect of β on the transfer $F(f)$ is a stretch in frequency proportional to $1/[1-\beta(\Delta t+f_0/k)] \approx 1+\beta(\Delta t+f_0/k)$ and a shift in frequency of $\beta(\Delta t_F f_0' + \dot{\ell} f_0^2/2ck) \approx \beta\Delta t_F f_0$ since $|\dot{\ell}|/c < 0.02$.

At this point a word about the validity of the approximation used above is in order. In the first place Eq. 18 must be obeyed if Eq. 17 is to be valid and hence Eqs. 23 and 24 hold. If the length of the target is L then $f(\sigma)$ must be zero for $\sigma > |L|$; therefore from Eq. 20 the maximum value of $|\alpha-\alpha_0|$ is (recall $\alpha_0 = 1-\dot{\ell}/c$) $|\beta \frac{L}{2} \delta(1-a)|$, whose maximum in turn is $L|\alpha|$. Thus for $f_0 = 3500$ Hz, $\beta = 220$ Hz, $T = 0.5$s and $L = 100$ m, Eq. 17 implies $|\alpha| < 3.82$ /s corresponding to a full turn in 1.57 minutes. If Eq. 6 is satisfied then the ordinary ambiguity function may be used to determine Γ^M_{FM} . The value of α of interest in Eq. 6 is $\alpha = \alpha_0$; hence $\alpha-1 = -\ell/c$, whose maximum is $(2v + v_T + v_R)/c$. Thus for the signal parameters just given $|2v + v_T + v_R| < 2.73$ kn to insure the ordinary ambiguity function may be always used — a rather severe constraint. On the other hand constraining $|2v + v_T + v|$ to be less than 60 kn, which is equivalent to $|\alpha-1| < 0.02$, does not seem very onerous. This constraint has been used previously in discussing approximations for γ^D_{FM} .

CONCLUSIONS

Two version of linear time-varying system theory are of use in analysis of active sonar systems: the ordinary (frequency-shift) version, which applies for CW signals and the doppler (frequency-stretch) version, which applies for FM signals. When the doppler version is used to determine the matched filter output for the bistatic echo from a moving, turning line target in a non-dispersive medium the result is that the modulation of the matched filter output is the time-shifted convolution of a frequency shifted and stretched impulse response representing the target and a window function representing the signal.

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APPENDICES

APPENDIX A

DERIVATION OF THE RELATIONSHIP BETWEEN THE IMPULSE RESPONSE AND THE SPREADING FUNCTIONS

Substitution of $y_I(t) = \delta(\tau-\tau')$ into Eq. 2a yields

$$h(t,t-\tau') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j\nu(t-\tau)}\delta(t-\tau-\tau')S(\nu,\tau)\, d\nu d\tau$$

$$= \int_{-\infty}^{-\infty}\int_{-\infty}^{\infty} e^{2\pi j\nu\tau'} S(\nu,t-\tau')\, d\nu$$

since $h(t,t-\tau')$ is the output due to an impulsive input at $t-(t-\tau') = \tau'$. Replacing $t-\tau'$ by $\tau (\Rightarrow \tau' = t-\tau)$ yields Eq. 3a. Similarly substitution of $y_I(t) = \delta(t-\tau')$ into Eq. 2b yields

$$h_{IO}(t,t-\tau') = \int_{-\infty}^{\infty}\int_{0}^{\infty} \delta[\alpha(t-\tau) -\tau']\, S_{IO}^{D}(\tau,\alpha)\, d\alpha d\tau$$

$$= \int_{0}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\alpha}\, \delta(\alpha t -\tau-\tau')\, S_{IO}^{D}(\frac{\tau}{\alpha},\alpha)\, dt d\alpha$$

$$= \int_{0}^{\infty} \frac{1}{\alpha}\, S_{IO}^{D}(\frac{\alpha t-\tau'}{\alpha}, \alpha)\, d\alpha$$

Replacing $t-\tau'$ by τ yields Eq. 3b.

APPENDIX B

DERIVATION OF THE FREQUENCY-SHIFT VERSION OF THE FUNDAMENTAL RELATIONSHIPS

The derivation takes place in two steps: first some properties of the modified double convolution $\underset{\tau,\nu}{\otimes}{}'$ are derived and then these properties are utilized to prove the relationships.

B.1 PROPERTIES OF THE MODIFIED DOUBLE CONVOLUTION

From the definition Eq. 5a

$$a \underset{\tau,\nu}{\otimes}{}' b(\tau,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j\nu'(\tau-t')} b(\tau-\tau',\nu-\nu')a(\tau',\nu')d\nu'd\tau'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j(\nu-\nu')t'} a(\tau-\tau',\nu-\nu')b(\tau',\nu')d\nu'd\tau',$$

which clearly does not in general equal $b \underset{\tau,\nu}{\otimes}{}' a(\tau,\nu)$. Furthermore, from Eq. 5a with τ,ν,τ' and ν' replaced by $\tau-\tau',\nu-\nu',\tau''-\tau'$ and $\nu''-\nu'$:

$$b \underset{\tau,\nu}{\otimes} a(\tau-\tau',\nu-\nu') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j(\nu''-\nu')(\tau-\tau'')} a(\tau-\tau'',\nu-\nu'')b(\tau''-\tau',\nu''-\nu')d\nu''d\tau''.$$

Substitution of this into Eq. 5a yields:

$$c \underset{\tau,\nu}{\otimes}{}' (b \underset{\tau,\nu}{x'} a)(\tau,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j\nu'(\tau-\tau')} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j(\nu''-\nu')(\tau-\tau'')}$$

$$\cdot\, a(\tau-\tau'',\nu-\nu'')b(\tau''-\tau,\nu''-\nu')d\nu''d\tau''\; c(\tau',\nu')d\nu'd\tau'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j\nu''(\tau-\tau'')} a(\tau-\tau'',\nu-\nu'')$$

$$\cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j\nu'(\tau''-\tau')} b(\tau''-\tau',\nu''-\nu')c(\tau',\nu')d\nu'd\tau'd\nu''d\tau''$$

$$= (c \underset{\tau,\nu}{\otimes}{}' b) \underset{\tau,\nu}{\otimes}{}' a(\tau,\nu) \quad ,$$

on use of Eq. 5a twice more.

Thus the operator $\underset{\tau,\nu}{\otimes}'$ has two properties of importance: it is associative, but not cummutative. The true double convolution operator $\underset{\tau,\nu}{\otimes}$, which differs from $\underset{\tau,\nu}{\otimes}'$ in that it is defined by Eq. 5a with the exponential term of the integrand removed, is both associative and commutative.

B.2 THE FUNDAMENTAL RELATIONSHIPS

Note the similarity in form between Eqs. 5a and 2a, the definition of the spreading function. The definition of the cross-ambiguity function (Eq. 1a) is seen to take the same form if it is rewritten

$$\phi_i(\tau,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j\nu'(\tau-\tau')}[\delta(\nu-\nu')\, x^*(\tau-\tau')]y_i(\tau)d\nu'd\tau'.$$

As a direct result both Eqs. 1a and 2a may be written in terms of $\underset{\tau,\nu}{\otimes}'$:

$$\phi_i(\tau,\nu) = y_i \underset{\tau,\nu}{\otimes}' \chi(\tau,\nu) \quad ,$$

$$y_0(\tau) = S_{I0} \underset{\tau,\nu}{\otimes}' y_I(\tau,\nu) \quad . \qquad \text{Eq B1)}$$

where

$$\chi(\tau,\nu) = x^*(\tau)\delta(\nu).$$

Use of these two results and the associative law yields:

$$\phi_0(\tau,\nu) = y_0 \underset{\tau,\nu}{\otimes}' \chi\,(\tau,\nu)$$

$$= (S_{I0} \underset{\tau,\nu}{\otimes}' y_I) \underset{\tau,\nu}{\otimes}' \chi(\tau,\nu)$$

$$= S_{I0} \underset{\tau,\nu}{\otimes}' (y_I \underset{\tau,\nu}{\otimes}' \chi)(\tau,\nu)$$

$$= S_{I0} \underset{\tau,\nu}{\otimes}' \phi_I(\tau,\nu),$$

which is the relationship of Fig. 2. Use of Eq. B1 twice and the associative law yields:

$$y_C(\tau) = S_{BC} \underset{\tau,\nu}{\otimes'} (S_{AB} \underset{\tau,\nu}{\otimes'} y_A)(\tau,\nu)$$

$$= (S_{BC} \underset{\tau,\nu}{\otimes'} S_{AB}) \underset{\tau,\nu}{\otimes} y_A(\tau,\nu).$$

But from Eq. B1,

$$y_C(\tau) = S_{AC} \underset{\tau,\nu}{\otimes'} y_A(\tau,\nu).$$

Since these results hold no matter what $y_A(\tau)$ is, it follows that the relationship of Fig. 3 must hold.

APPENDIX C

DERIVATION OF THE FREQUENCY STRETCH VERSION OF THE FUNDAMENTAL RELATIONSHIPS

As with the frequency-shift version, the derivation takes place in two steps: first some properties of $\underset{\tau,\alpha}{\otimes'}$ are derived and then these properties are utilized to prove the relationships.

C.1 PROPERTIES OF THE MODIFIED DOUBLE CONVOLUTION

From the definition Eq. 5b and an interchange of order of integration:

$$a^D \underset{\tau,\alpha}{\otimes'} b^D(\tau,\alpha) = \int_0^{\infty}\int_{-\infty}^{\infty} b^D[\alpha'(\tau-\tau'), \frac{\alpha}{\alpha'}]\, a^D(\tau',\alpha')d\tau'\, d\alpha'$$

$$= \int_0^{\infty}\int_{-\infty}^{\infty} \frac{1}{\alpha'}\, a^D(\tau - \frac{\alpha'\tau'}{\alpha}, \frac{\alpha}{\alpha'})\, b^D(\tau',\alpha')d\tau'\, d\alpha' \quad ,$$

which clearly does not in general equal $b^D \underset{\tau,\alpha}{\otimes'} a^D(\tau,\alpha)$. Furthermore, from Eq. 5b with τ,α,τ' and α' replaced by $\alpha'(\tau-\tau'), \frac{\alpha}{\alpha'}, \alpha'(\tau''-\tau')$ and $\frac{\alpha''}{\alpha'}$:

$$b^D \underset{\tau,\alpha}{\otimes'} a^D[\alpha'(\tau-\tau'), \frac{\alpha}{\alpha'}] = \int_{-\infty}^{\infty}\int_0^{\infty} a^D[\alpha''(\tau-\tau''), \frac{\alpha}{\alpha''}]\, b^D[\alpha'(\tau''-\tau'), \frac{\alpha''}{\alpha}]d\alpha''\, d\tau''$$

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Substitution of this into Eq. 5b yields:

$$c^D \underset{\tau,\alpha}{\otimes}' (b^D \underset{\tau,\alpha}{\otimes}' a^D)(\tau,\alpha) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} a^D[\alpha''(\tau-\tau''), \frac{\alpha}{\alpha''}] b^D[\alpha'(\tau''-\tau'), \frac{\alpha''}{\alpha'}] d\alpha'' \, d\tau''$$

$$. \; c^D(\tau',\alpha') d\alpha' \, d\tau'$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} a^D[\alpha''(\tau-\tau''), \frac{\alpha}{\alpha''}] \int_{-\infty}^{\infty}\int_{0}^{\infty} b^D[\alpha'(\tau''-\tau'), \frac{\alpha''}{\alpha'}]$$

$$. \; c^D(\tau',\alpha') d\alpha' d\tau' d\alpha'' d\tau''$$

$$= (c^D \underset{\tau,\alpha}{\otimes}' b^D) \underset{\tau,\alpha}{\otimes}' a^D(\tau,\alpha) \quad ,$$

on use of (5b) twice more. Again $\underset{\tau,\alpha}{x}'$ is associative but not commutative.

C.2 THE FUNDAMENTAL RELATIONSHIP

Note the similarity in form between (5b) and (2b), the definition of the doppler spreading function. The definition of the doppler cross-ambiguity function (1b) is seen to take the same form if it is rewritten

$$\phi_i^D(\tau,\alpha) = \int_{-\infty}^{\infty}\int_{0}^{\infty} \delta(\frac{\alpha-\alpha'}{\alpha}) \; x^*[\tau-\tau')] \;\; y_i(\tau') \; d\alpha' \; d\tau'$$

(The factor α in front of the integral in the original definition (1b) was inserted to yield this convenient form.) As a direct result both (1b) and (2b) may be written in terms of $\underset{\tau,\alpha}{\otimes}'$:

$$\phi_i^D(\tau,\alpha) = y_i \underset{\tau,\alpha}{\otimes}' \chi^D(\tau,\alpha),$$

$$y_0^D(\tau) = S_{IO}^D \underset{\tau,\alpha}{\otimes}' \;\; y_I(\tau,\alpha), \qquad \text{(Eq. C1)}$$

where

$$\chi^D(\tau,\alpha) = x^*(\tau)\delta(\frac{\alpha-1}{\alpha}).$$

Use of these two results and the associative law yields:

$$\phi_0^D(\tau,\alpha) = y_0 \underset{\tau,\alpha}{\times}{}' \chi^D(\tau,\alpha)$$

$$= (S_{IO}^D \underset{\tau,\alpha}{\otimes}{}' y_I) \underset{\tau,\alpha}{\otimes}{}' \chi^D(\tau,\alpha)$$

$$= S_{IO}^D \underset{\tau,\alpha}{\otimes}{}'(y_I \underset{\tau,\alpha}{\otimes}{}' \chi^D)(\tau,\alpha)$$

$$= S_{IO}^D \underset{\tau,\alpha}{\otimes}{}' \phi_I^D(\tau,\alpha) \quad ,$$

which is the relationship of Fig. 2. Use of (B-1) twice and the associative law yields:

$$y_C(\tau) = S_{BC}^D \underset{\tau,\alpha}{\otimes}{}'(S_{AB}^D \underset{\tau,\alpha}{\otimes}{}' y_A)\ (\tau,\alpha)$$

$$= (S_{BC}^D \underset{\tau,\alpha}{\otimes}{}' S_{AB}^D) \underset{\tau,\alpha}{\otimes}{}' y_A(\tau,\alpha)$$

But also from (C-1),

$$y_C(\tau) = S_{AC}^D \underset{\tau,\alpha}{\otimes}{}' y_A(\tau,\alpha).$$

Since these results hold no matter what $y_A(\tau)$ is, it follows that the relationship of Fig. 3 must hold.

APPENDIX D

DERIVATION OF THE APPROXIMATE RELATIONSHIPS

This appendix is concerned with showing two things: For low BT product signals, such as CW pulse, the spreading function of a moving target may be replaced by its much simpler doppler spreading function and the standard version of the theory applied. For high BT product signals, such as linear FM pulses, the doppler spreading functions of the medium may be replaced by its ordinary spreading functions and the doppler version of the theory applied.

D.1 REPLACEMENT OF THE ORDINARY SPREADING FUNCTION BY THE DOPPLER SPREADING FUNCTION

A fundamental approximation used in communications theory is that for a small BT signal modulated on a carrier $x(t)$ the doppler shift affects only the carrier:

$$x(\alpha t) \approx e^{2\pi j(\alpha-1)f_0 t}\, x(t) \,. \qquad \text{(Eq. D1)}$$

This approximation is discussed in detail by Rihaczek (*[1]* pp 56-65), where he derives the general condition that for it to be a good approximation (6) must be satisfied.

When the approximation is valid its substitution into (1b) yields

$$\phi_i^D(\tau,\alpha) \approx \alpha \int_{-\infty}^{\infty} e^{-2\pi j(\alpha-1)\, f_0(t-\tau)}\, x^*(t-\tau)\, y_i(t)\, dt \quad ,$$

which on comparison with (1a) becomes

$$\phi_i^D(\tau,\alpha) \approx \alpha\phi_i[\tau,(\alpha-1)\, f_0] .$$

Substitution of this result into the fundamental relationship given by Fig. 2 and (5b) yields:

$$\phi_0[\tau,\, (\alpha-1)\, f_0] \approx \frac{1}{\alpha} \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\alpha}{\alpha'} \cdot \phi_I[\alpha'(\tau-\tau'),\, (\frac{\alpha}{\alpha'}-1)f_0]\, S_{0I}^D(\tau',\alpha')d\alpha'\, d\tau'$$

$$\approx \int_{-\infty}^{\infty}\int_0^{\infty} e^{2\pi j(\alpha'-1)f_0(\tau-\tau')}\, \phi_I[\tau-\tau',\, (\alpha-\alpha')f_0]S_{I0}^D(\tau',\alpha')d\alpha' d\tau'$$

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Since from (1a) and (D-1)

$$\phi_I(\alpha'\tau,\nu) = \int_{-\infty}^{\infty} e^{-2\pi j\nu(t-\alpha'\tau)}\, x^*(t-\alpha'\tau)\, y_i(t)\, dt$$

$$= \alpha' \int_{-\infty}^{\infty} e^{-2\pi j\nu\alpha'(t-\tau)}\, x^*[\alpha'(t-\tau)]\, y_i(\alpha' t)\, dt$$

$$\approx \alpha' e^{2\pi j(\alpha'-1)f_o\tau} \int_{-\infty}^{\infty} e^{-2\pi j\alpha'\nu(t-\tau)}\, x^*(t-\tau)\, y_I(t)\, dt$$

$$= \alpha'\, e^{2\pi j(\alpha'-1)f_o\tau}\, \phi_I(\tau,\alpha'\nu).$$

Comparison of the above equation for $\phi_0[\tau,(\alpha-1)f_o]$ with the fundamental relationship given by Fig. 2 and (5a) yields (7).

D.2 REPLACEMENT OF THE DOPPLER SPREADING FUNCTION BY THE ORDINARY SPREADING FUNCTION

An obvious extension of D-1 is

$$x(\alpha' t) \approx e^{2\pi j\,(\alpha'-\alpha)f_o t}\, x(\alpha t), \qquad |\alpha'-\alpha| < \frac{0.2}{BT}\,. \qquad \text{(Eq. D2)}$$

To derive (8) from this result note its substitution into (1b)

$$\frac{1}{\alpha + \frac{\nu}{f_o}}\, \phi_I^D(\tau,\alpha + \frac{\nu}{f_o}) \approx \int_{-\infty}^{\infty} e^{2\pi j\nu(t-\tau)}\, x^*[\alpha(t-\tau)]\, y_I(t)\, dt$$

for $\nu < 0.2\, f_o/BT$. The right side of this equation is the ordinary cross-ambiguity function between $x(\alpha t)$ and $y_I(t)$. Furthermore, from (1b), $\phi_0^D(\tau,\alpha)$ is the ordinary cross-ambiguity function between $x(\alpha t)$ and $y_0(t)$ evaluated at $\nu = 0$; therefore substitution of $\phi_0^D(\tau,\alpha)$ for $\phi_0(\tau,0)$ and $\frac{1}{\alpha-\frac{\nu}{f_o}}\, \phi_I^D(\tau-\tau',\alpha-\frac{\nu'}{f_o})$ for $\phi_I(\tau-\tau',-\nu')$ in the fundamental relationship given in Fig. 2 and (5a) with $\nu = 0$ is permitted and yields (8).

APPENDIX E

DERIVATION OF APPROXIMATE FORMS OF THE LINEAR FM DOPPLER AMBIGUITY FUNCTION

Three approximations are to be derived in this appendix: The first two are approximations to the window function γ_{FM}^{M} from which the FM doppler ambiguity function is readily obtainable via (11b). The last approximation is of $\gamma_{FM}^{D'}(\alpha\tau,\ 1/\alpha)$, the doppler ambiguity function of an FM pulse centered at t_T rather than 0, in terms of $\gamma_{FM}^{D}(\alpha_o\tau,\ 1/\alpha_o)$ for α near α_o.

E.1 THE WINDOW FUNCTION

From the definition of the doppler ambiguity (Ib)

$$\gamma_{FM}^{D}(\alpha\tau,\ 1/\alpha) = \frac{1}{\alpha}\int_{-\infty}^{\infty} x_{FM}^{*}\left(\frac{t}{\alpha} - \tau\right) x_{FM}(t)\ dt$$

$$= \int_{-\infty}^{\infty} x_{FM}^{*}(t-\tau+\tau')\ x_{FM}[\alpha(t+\tau')]\ dt$$

after change of dummy variable. Substitution of this result and the definition of the FM pulse x_{FM} into (12) (with α_o replaced by α) yields

$$\gamma_{FM}^{M}\left(\tau + \frac{\alpha+1}{\alpha}\frac{f_o}{k}, \alpha\right) = \frac{1}{T}\, u(t_{MAX}-t_{MIN}) \int_{t_{MIN}}^{t_{MAX}} e^{2\pi j\phi_o}\ dt \qquad \text{(Eq. E1)}$$

where

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$$\phi_o = -f_o(t-\tau+\tau') - \frac{1}{2}k(t-\tau+\tau')^2 + \alpha f_o(t+\tau') + \frac{1}{2}k\alpha^2(t+\tau') - \frac{\alpha+1}{2} f_o\tau$$

$$= (\alpha-1)\ f_o\ \left(t - \frac{\tau}{2} + \tau'\right) + \frac{1}{2}k\alpha^2(t+\tau')^2 - \frac{1}{2}k(t-\tau+\tau')^2$$

and, since $x_{FM}(t-\tau+\tau')$ is zero outside the interval $[\tau-\tau'-\frac{T}{2}, \tau-\tau'+\frac{T}{2}]$ and $x_{FM}[\alpha(t+\tau')]$ is zero outside the interval $[-\tau'-\frac{T}{2\alpha}, -\tau'+\frac{T}{2\alpha}]$,

$$t_{MAX} = \min[\tau-\tau'+\frac{T}{2}, -\tau'+\frac{T}{2\alpha}] = \frac{\tau}{2}-\tau'+\frac{T}{2}+\min[\frac{\tau}{2}, -\frac{\tau}{2}-\frac{T(\alpha-1)}{2\alpha}] ,$$

$$t_{MIN} = \max[\tau-\tau'-\frac{T}{2}, -\tau'-\frac{T}{2\alpha}] = \frac{\tau}{2}-\tau'-\frac{T}{2}+\max[\frac{\tau}{2}, -\frac{\tau}{2}+\frac{T(\alpha-1)}{2\alpha}] .$$

If τ' is chosen to eliminate the terms of ϕ_o that are of first order in t then γ_{FM}^{M} can be easily written in terms of Fresnel's integrals. Rather than proceed directly on this track it is appropriate to make some approximations.

Consider first the following approximate limits:

$$t'_{MAX} = \frac{\tau}{2} - \tau' + \frac{T}{2} - |\frac{\tau}{2}| ,$$

$$t'_{MIN} = \frac{\tau}{2} - \tau' - \frac{T}{2} + |\frac{\tau}{2}| .$$

The maximum error between the actual and approximate limits is $T|\alpha-1|/\alpha$; furthermore the integrand of (E-1) has unity magnitude; therefore, since $u(t'_{MAX} - t'_{MIN}) = \sqcap(\tau,T)$, (E-1) may be rewritten

$$\gamma_{FM}^{M}(\tau+\frac{\alpha+1}{\alpha}\frac{f_o}{k},\alpha) = \frac{1}{T}\sqcap(\tau,T)\int_{t'_{MIN}}^{t'_{MAX}} e^{+2\pi j\phi_o}\,dt + \varepsilon_1 ,$$

where

$$|\varepsilon_1| < \frac{1}{T}\cdot\frac{T|\alpha-1|}{\alpha} = |\alpha-1|/\alpha = v|y|/(1+v\ y)$$

from (15).

The next approximation is to neglect $(\alpha-1)^2$ terms in ϕ ; i.e. to approximate $\alpha^2 = (\alpha-1+1)^2 = (\alpha-1)^2 + 2(\alpha-1) + 1$ by $2(\alpha-1) + 1 = 2\alpha-1$. If ϕ_o' is ϕ_o with this approximation, clearly

$$\phi_o = \phi_o' + \tfrac{1}{2}k(\alpha-1)^2(t+\tau')^2 .$$

Furthermore, since $|e^{jx}-1|<x$ for real x, (E-1) may rewritten

$$\gamma_{FM}^{M}(\tau+\frac{\alpha+1}{\alpha}\frac{f_o}{k},\alpha) = \frac{1}{T}\sqcap(\tau;T)\int_{t'_{MIN}}^{t'_{MAX}} e^{-2\pi j\phi_o'}\,dt + \varepsilon_1 + \varepsilon_2$$

where

$$|\varepsilon_2| \leqslant \frac{1}{k} \sqcap(\tau;T) \int_{t'_{MIN}}^{t'_{MAX}} \pi|k|\;(\alpha-1)^2(t+\tau')^2 \;.$$

The value of this last integral is independent of τ' ; it is convenient to select $\tau' = \tau$ for its evaluation :

$$|\varepsilon_2| \;\leqslant\; \frac{1}{T} \sqcap(\tau;T)\, \frac{\pi}{3}\, |k|(\alpha-1)^2\; t^3 \Bigg|_{-\frac{T-|\tau|-\tau}{2}}^{\frac{T-|\tau|+\tau}{2}}$$

$$= \frac{1}{T} \sqcap(\tau;T)\, \frac{\pi}{3}\, |k|(\alpha-1)^2 \; \left[(\tfrac{T}{2})^3 + (\tfrac{T}{2} - |\tau|)^3 \right]$$

$$\leqslant \frac{\pi}{12}\; |k|\; T^2\; (\alpha-1)^2 < 0.27\; BT(\alpha-1)^2 = 0.27\; v\; x\; y^2$$

from (15). Therefore for $\varepsilon = \varepsilon_1 + t_2$

$$|\varepsilon| \leqslant |\varepsilon_1| + |\varepsilon_2| \leqslant |\alpha-1|(\frac{1}{\alpha} + 0.27\; BT\; |\alpha-1|) = v|y|(\frac{1}{1+vy} + 0.27|x\; y|) \;,$$

as is given in (16).

Now τ' is selected so that the multiplier of t in ϕ_o is zero; i.e. so that

$$\frac{\partial \phi_o'}{\partial t}\Big|_{t=0} = (\alpha-1)\; f_o + (2\alpha-1)\; k\; \tau' \;-\; k(\tau'-\tau) = 0 \;. \qquad \text{(Eq. E2)}$$

If this is multiplied by $(t - \frac{\tau}{2} + \tau')$ and subtracted from ϕ_o' , the result is

$$\phi_o' = (\alpha-\; 1)\; k\; t^2 + \frac{2\alpha-1}{2}\, k[(\tau')^2 - \tau'(2\tau'-\tau)] - \frac{1}{2}\, k\lfloor(\tau'-\tau)^2 - (\tau'-\tau)(2\tau'-\tau)\rfloor$$

$$= (\alpha-1)k\; t^2 - (\alpha-1)k\; \tau'(\tau'-\tau) \;.$$

But from Eq. E.2

$$2(\alpha-1)k\; (\tau' - \frac{\tau}{2}) = \alpha k\tau + (\alpha-1)\; f_o \quad ;$$

therefore

$$\phi_o' = (\alpha-1)\, k\, t^2 - \frac{[\alpha k\tau + (\alpha-1)f_o]^2}{4(\alpha-1)k} + \frac{(\alpha-1)^k}{4}\,\tau^2$$

and (E.1) becomes (on replacing $\tau + \frac{\alpha-1}{\alpha}\frac{f_o}{k}$ by τ and $2\sqrt{|(\alpha-1)k|}\ t$ by t)

$$\gamma_{FM}^{M}(\tau,\alpha) = \frac{1}{T}\sqcap\left(\tau - \frac{\alpha-1}{\alpha}\frac{f_o}{k}, T\right)\frac{e^{-\frac{\pi}{2}j\phi}}{2\sqrt{|(\alpha-1)k|}}\int_a^b e^{\frac{1}{2}\tau j\,\sin[(\alpha-1)k]t}\,dt + \epsilon$$

where

$$\phi = \frac{(\alpha k\tau)^2}{(\alpha-1)k} - (\alpha-1)k\left(\tau - \frac{\alpha-1}{\alpha}\frac{f_o}{k}\right)^2 ,$$

$$a = \sqrt{(\alpha-1)k}\left[\frac{\alpha k\tau}{(\alpha-1)k} + T - \left|\tau - \frac{\alpha-1}{\alpha}\frac{f_o}{k}\right|\right],$$

$$b = \sqrt{(\alpha-1)k}\left[\frac{\alpha k\tau}{(\alpha-1)k} - T + \left|\tau - \frac{\alpha-1}{\alpha}\frac{f_o}{k}\right|\right] .$$

This result is equivalent to Eq. 16 on substitution of Eq. 15.

When BT is large an even simpler approximation may be found. As noted in the main text the Fourier transform of $\gamma_{FM}^{D}(\alpha\tau,1/\alpha)$ is the product of the matched filter transfer function and the Fourier transform of a frequency stretched linear FM pulse. But for large BT (see Rihaczek [1], Chapter 7) the frequency stretched linear FM has Fourier transform

$$\frac{1}{\alpha\sqrt{B}}\sqcap(f - \alpha f_o\ ;\ \alpha B/2)\, e^{-\pi j\left[\frac{1}{\alpha^2 k}(f-\alpha f_o)^2 - \sin(k)/4)\right]} .$$

Furthermore, since the matched filter is matched to the unstretched signal it has transfer function

$$\frac{1}{\sqrt{B}}\sqcap(f-f_o;B/2)e^{\pi j\left[\frac{1}{k}(f-f_o) - \sin(k)/4\right]}$$

Taking their product yields

$$\Gamma^{D}_{FM}(f,\ 1/\alpha) = \frac{1}{\alpha B} \quad \frac{\alpha-1}{2} f_o ; \frac{\alpha+1}{4} B \qquad (f - \frac{\alpha+1}{2} f_o - \frac{|\alpha-1|}{4} B ; \frac{\alpha+1}{4} B - \frac{|\alpha-1|}{2} f_o$$

$$\cdot\ e^{\pi j \frac{1}{k} \left(\frac{\alpha^2-1}{\alpha^2} f^2 - \frac{2(\alpha-1)}{\alpha} f f_o \right)} .$$

To obtain Γ^{M}_{FM} , it is apparent from Eq. 12 that $\Gamma^{D}_{FM}(f,\ 1/\alpha)$ must be frequency shifted by $-\frac{\alpha+1}{2} f_o$ and then time shifted by $-\frac{\alpha-1}{\alpha} \frac{f_o}{k}$. Since in the frequency domain a time shift of $\Delta\tau$ is a multiplication by $e^{2\pi j\ f\Delta\tau}$, the result is

$$\Gamma^{D}_{FM}(f,\ 1/\alpha) = \frac{1}{\alpha B} \quad \frac{\alpha-1}{2} f_o ; \frac{\alpha+1}{4} B \qquad f - \frac{|\alpha-1|}{4} B ; \frac{\alpha+1}{4} B - \frac{|\alpha-1|}{2} f_o$$

$$\cdot\ e^{\pi j \frac{(\alpha^2-1)}{k\alpha^2} \left(f^2 + \frac{(\alpha-1)^2}{\alpha+1} f f_o + \frac{1}{4} (\alpha-1)^2 f_o^2 \right)}$$

$$= \frac{1}{(1+vy)B} \quad \frac{4}{x} ; \frac{2+vy}{2} \qquad 2w - v \frac{|xy|}{2x} ; \frac{2+vy}{2} - |y/x|$$

$$\cdot\ e^{\pi j (2+vy) xy \left(w^2 + \frac{vwy^2}{(2+vy)x} + \frac{1}{4} \frac{y}{x}^2 \right)}$$

on use of (15). Setting $v = 0$ yields (17) except for the factor $1/B$, whose presence was explained earlier.

The Fourier transform of $|\gamma^{M}_{FM}(z,y;x,o)|$ with respect to z is from Fourier transform theory just $\Gamma^{M*}_{FM}(-w,y;x,o) \underset{w}{x} \Gamma^{M}_{FM}(w,y;x,o)$ where $\underset{w}{x}$ signifies convolution with respect to w ; i.e. it is the correlation of Γ^{M}_{FM} with itself. Use of (17) yields

$$\Gamma^{M}_{FM}(-w,y;x,o) \underset{w}{x} \Gamma^{M}_{FM}(w,y;x,o) = \int_{-\infty}^{\infty} \Gamma^{M*}_{FM}(w'y;x,o)\ \Gamma^{M}_{FM}(w+w',y;x,o)\ dw' ,$$

$$= \sqcap(y/x;1) \int_{-\infty}^{\infty} (2w';1-|\tfrac{y}{x}|) \quad [2(w+w');1-|\tfrac{y}{x}|]\ e^{2\pi jxy[(w+w')^2-(w')^2]} dw' .$$

If $o < w < 1 - |\frac{y}{x}|$,

$$(2w';1-|\tfrac{y}{x}|) \quad [2(w+w');1- |\tfrac{y}{x}|] = u(2w'+1-|\tfrac{y}{x}|)-u[2(w'+w)-1-|\tfrac{y}{x}|]$$

and the integral becomes

$$\int_{-(1-|\frac{y}{x}|)/2}^{(1-|\frac{y}{x}|-w)/2} e^{2\pi jxy[(w+w')^2-(w')^2]}\, dw' = e^{2\pi jxyw^2} \frac{e^{4\pi jxyww'}}{4\pi jxyw} \Bigg|_{-(1-|\frac{y}{2}|)/2}^{(1-|\frac{y}{x}|-2w)/2}$$

$$= \frac{e^{2\pi jxy[(1-|\frac{y}{x}|)w-w^2]} - e^{-2\pi jxy[(1-|\frac{y}{x}|)w-w^2]}}{4\pi jxyw}$$

$$= (1-|\tfrac{y}{x}|-w) \sin 2\pi xy[(1-|\tfrac{y}{x}|)w-w^2] \quad ,$$

which agrees with (18). The same procddure may be carried out for $-1+|\frac{y}{x}| < w < 1$ to obtain the complete result.

E.2 THE AMBIGUITY FUNCTION FOR TIME SHIFTED LINEAR FM PULSES

From (D-1) for $|\alpha-\alpha_o| < 0.2/BT$

$$x_{FM}(\alpha t) \approx e^{2\pi\, j(\alpha-\alpha_o)f_o t}\, x_{FM}(\alpha_o t) \quad ,$$

since the carrier of x_{FM} is f_o . On the other hand, by the definition of x_{FM}, (9)

$$x_{FM}(\alpha_o t + \frac{\alpha-\alpha_o}{\alpha_o k}) \qquad e^{2\pi j\left[\frac{\alpha-\alpha_o}{\alpha_o k} f_o^2 + (\alpha-\alpha_o)f_o t\right]}\, x_{FM}(\alpha_o t)$$

provided that $(\alpha_o t + \frac{\alpha-\alpha_o}{\alpha_o k} f_o;T/2) \quad (\alpha_o t;T/2)$ and $e^{\pi j} \frac{\alpha-\alpha_o}{\alpha_o^2 k} f_o^2$

is negligible. The former will be true if $\frac{|\alpha-\alpha_o|}{\alpha_o B} f_o T \ll T$ and the

latter will be true if $\frac{(\alpha-\alpha_o)^2}{\alpha_o^2 B} f_o^2 T \pi \ll \frac{\pi}{2}$ these two conditions and

the other above will be satisfied if (20) of the main text is satisfied. Comparison of the two results above yields

$$x_{FM}(\alpha t) \quad e^{-2\pi j \frac{\alpha-\alpha_o}{\alpha} \frac{f_o^2}{k}} x_{FM}(\alpha_o t + \frac{\alpha-\alpha_o}{\alpha_o k} f_o) \quad . \tag{Eq. E.3}$$

On the other hand by definition (see (1b))

$$\gamma_{FM}^{D'}(\alpha\tau,1/\alpha) = \int_{-\infty}^{\infty} x_{FM}^{*} (t-\tau-t_T) x_{FM}(\alpha t-t_T)\, dt \quad . \tag{Eq. E.4}$$

$$= \int_{-\infty}^{\infty} x_{FM}^{*}[t-\tau- \frac{\alpha-1}{\alpha} t_T]\, x_{FM}(\alpha t)\, dt \ .$$

Substitution of Eq. E.3 into Eq. E.4 yields

$$\gamma^{D}(\alpha\tau,1/\alpha) \quad e^{-2\pi j \frac{\alpha-\alpha_o}{\alpha_o} \frac{f_o^2}{k}} \int_{-\infty}^{\infty} x_{FM}(t-\tau- \frac{\alpha-1}{\alpha} t_T)\, x_{FM}[\alpha_o(t+\frac{\alpha-\alpha_o}{\alpha_o^2 k} f_o)]dt$$

$$= e^{-2\pi j \frac{\alpha-\alpha_o}{\alpha_o} \frac{f_o^2}{k}} \int_{-\infty}^{\infty} x_{FM}(t-\tau- \frac{\alpha-\alpha_o}{\alpha_o^2 k} f_o - \frac{\alpha-1}{\alpha} t_T)\, x_{FM}(\alpha_o t)\, dt$$

after a change of dummy variable. This result in turn yields (19) on comparison with the equation for $\gamma_{FM}^{D}(\alpha_o t,\ 1/\alpha_o)$ found by setting $t_T = 0$ and $\alpha = \alpha_o$ in Eq. E.4.

APPENDIX F

DERIVATION OF THE BISTATIC DOPPLER SPREADING FUNCTION OF A MOVING, TURNING LINE TARGET

The derivation takes place in three parts: First the spreading functions to and from a point target with constant range rate are determined. Next using these results and trigonometry the spreading function of a point with unit reflectivity on a moving, turning line target is found. Finally these points are summed weighted by appropriate reflectivity to obtain the overall spreading function.

F.1 THE SPREADING FUNCTION TO AND FROM A POINT WITH CONSTANT RANGE RATE

Consider a point whose range ρ is given by

$$\rho(t) = \rho_T + \dot{\rho}_T\, t,$$

then for transmission to the point

$$y_0(t) = y_I\left(t - \frac{\rho_T}{c} - \frac{\dot{\rho}_T}{c}\, t\right) .$$

Consider also the spreading function

$$S_1^D(\tau,\alpha;\ \tau_1,\alpha_1) \triangleq \delta(\tau-\tau_1)\delta(\alpha-\alpha_1)$$

From (2b)

$$y_o(t) = y_I[t-\alpha_1\tau_1 + (\alpha_1-1)t] \qquad \text{(Eq. F.1)}$$

Comparison of the two equations relating y_o and y_I shows that

$$S_{TP}^D(\tau,\alpha;\rho_T,\dot{\rho}_T) = \delta\,(\tau-\tau_{TP})\delta\,(\alpha-\alpha_{TP}) \quad , \qquad \text{(Eq. F.2)}$$

where

$$\alpha_{TP} \triangleq 1-\dot{\rho}_T/c,\ \tau_{TP} \triangleq \frac{\rho_T}{c-\dot{\rho}_T} \quad \frac{\rho_T}{c} \quad ,$$

is the Doppler Spreading function of transmission to a moving point target.

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Clearly for reception from the point (with ρ_T and $\dot{\rho}_T$ replaced by ρ_R and $\dot{\rho}_R$)

$$y_0\left(t' + \frac{\rho_R + \dot{\rho}_R t'}{c}\right) = y_I(t') \quad .$$

Replacing t' by $t = t' + \dfrac{\rho_R + \dot{\rho}_R t'}{c}$ yields

$$y_0(t) = y_I\{t - (1 + \frac{\dot{\rho}_R}{c})^{-1} \frac{\rho_R}{c} - [(1 + \frac{\dot{\rho}_R}{c}) - 1]\, t \quad .$$

Comparison of this result with F.1 shows that the spreading function is

$$S^D_{PR}(\tau,\alpha;\rho_R,\dot{\rho}_R) = \delta(\tau-\tau_{PR})\delta(\alpha-\alpha_{PR}) \;, \qquad \text{(Eq. F.3)}$$

where

$$\alpha_{PR} \triangleq (1 + \frac{\dot{\rho}_R}{c})^{-1} \approx 1 - \frac{\dot{\rho}_R}{c} \;, \; \tau_{PR} \triangleq \rho_R/c.$$

F.2 THE SPREADING FUNCTION TO A POINT ON A MOVING, TURNING TARGET

Application of the law of cosines to Fig. 9 yields

$$\rho_T^2 = r_T^2 + \sigma^2 - 2\, r_T\, \sigma \cos\alpha \;,$$

$$\rho_R^2 = r_R^2 + \sigma^2 - 2\, r_R\, \sigma \cos(\alpha-\gamma) \;.$$

But $\sigma << r_T$ and r_R (on the order of 10^2m compared with on the order of 10^4m to 10^5m); therefore

$$\rho_T = r_T \sqrt{1 - 2\frac{\sigma}{r_T}\cos\alpha + \frac{\sigma^2}{r_T^2}} \quad r_T - \sigma\cos\alpha \;,$$

$$\rho_R \approx r_R - \sigma\cos(\alpha-\gamma). \qquad \text{(Eq. F.4)}$$

Differentiation of these results yields

$$\dot{\rho}_T = \dot{r}_T + \sigma \sin\alpha\, \dot{\alpha},$$

$$\dot{\rho}_R = \dot{r}_R + \sigma \sin(\alpha-\gamma)\,(\dot{\alpha}-\dot{\gamma}) \;, \qquad \text{(Eq. F.5)}$$

where from Fig. 9

$$\dot{r}_T = -(v \cos\alpha + v_T \cos \psi_T) \quad ,$$

$$\dot{r}_R = -[v \cos(\alpha-\gamma) + v_R \cos \psi_R] \ .$$

Now consider the doppler spreading function $S_P^D(\tau,\alpha;\sigma)$ of a point at location σ on the target assumed to have unity reflectivity.

If the point is assumed to have ranges $\rho_T + \dot{\rho}_T t$ and $\rho_R + \dot{\rho}_R t$ from the transmitter and receiver respectively Fig. 3 may be used to obtain S_P ; therefore Δt_T, which is the time of arrival of the signal at the target, must be selected sufficiently small in magnitude that these assumptions are valid.

Use of the fundamental relationship given by Fig. 3 and (5b) on Eqs. F.2 and F.3 yields

$$\begin{aligned} S_P^D(\alpha,\tau;\sigma) &= \int_{-\infty}^{\infty}\int_{0}^{\infty} \delta(\tfrac{\alpha}{\alpha'}-\alpha_{TP})\delta(\alpha'(\tau-\tau')-\tau_{TP})\delta(\alpha'-\alpha_{PR})\delta(\tau'-\tau_{PR})d\alpha' d\tau' \\ &= \delta(\frac{\alpha}{\alpha_{PR}} - \alpha_{TP})\ \delta[\alpha_{PR}(\tau-\tau_{PR}) - \tau_{TP}] \\ &= \delta(\alpha-\alpha_{PR}\ \alpha_{TP})\delta(\tau-\tau_{PR}-\frac{\tau_{TP}}{\alpha_{PR}}) = \delta[\alpha-\alpha_P(\sigma)]\delta[\tau-\tau_P(\sigma)] \ , \end{aligned}$$

since $a\delta(at) = \delta(t)$.

It is convenient to approximate $\alpha_P(\sigma)$ and $\tau_P(\sigma)$. From (F.2) and (F.3)

$$\alpha_P(\sigma) = (1-\frac{\dot{\rho}_T}{c})/(1+\frac{\dot{\rho}_R}{c}) \ .$$

Since $\dot{\rho}_T$ and $\dot{\rho}_R << c$ (two orders of magnitude at least)

$$\begin{aligned} \alpha_P(\sigma) &\approx 1 - (\dot{\rho}_T + \dot{\rho}_R)/c \quad , \\ &= 1 - \{\dot{r}_T + \dot{r}_R + \sigma\ [\sin\alpha\ \dot{\alpha} + \sin(\alpha-\gamma)\ (\dot{\alpha}-\dot{\gamma})]\}\ /c \quad , \\ &= 1 - (\dot{\ell} + \beta\ \delta\ \sigma)/c \quad , \qquad \text{(Eq. F.6)} \end{aligned}$$

from (F.5) and (23). Also from (F.2) and (F.3)

$$\tau(\sigma) = \frac{\left(1+\frac{\dot{\rho}_R}{c}\right)}{\left(1-\frac{\dot{\rho}_T}{c}\right)} \frac{\rho_T}{c} + \frac{\rho_R}{c}$$

$$= \frac{\rho_T}{\alpha_\rho(\sigma)c} + \frac{\rho_R}{c} = \frac{\rho_{T}+\rho_R}{c} - \frac{\alpha_\rho(\sigma)-1}{\alpha_\rho(\sigma)} \frac{\rho_T}{c}$$

$$= \frac{r_T+r_R-\sigma[\cos\alpha + \cos(\alpha-\gamma)]}{c} - \frac{\alpha_p(\sigma)-1}{\alpha_p(\sigma)} \frac{r_T}{c} + \frac{\alpha(T)-1}{\alpha_p(\sigma)} \frac{\sigma}{c} \cos\alpha$$

from (F.4). But

$$\frac{\alpha_p(\sigma)-1}{\alpha_p(\sigma)} = 1 - \frac{1}{\alpha_p(\sigma)} \approx 1 - 1 + \frac{\dot{\ell}}{c} + \ldots$$

Since σ is small compared to r_T the higher order terms in this expansion may be neglected in the third term of $\tau(\sigma)$ to yield with the aid of (23)

$$\tau_p(\sigma) \approx [(\ell-\delta\sigma)/c] - (r_T/c) \frac{\alpha_p(r)-1}{\alpha_p(\sigma)} + a\delta\sigma/c \quad ,$$

F.3 THE OVERALL SPREADING FUNCTION

Now let the reflectivity of each point σ be $f(\sigma)$; then from linearity the spreading function of the target is

$$S^D(\tau,\alpha) = \int_{-\infty}^{\infty} f(\sigma)\, S_p(\tau,\alpha;\, \sigma)\, d\sigma$$

This integration must be performed for two different cases: $\beta = o$ and $\beta \neq o$. For $\beta = o$, $\alpha_p(\sigma) = 1 - \dot{\ell}/c$ and is independent of σ from (F.6); hence

$$S(\tau,\alpha) \approx \int_{-\infty}^{\infty} f(\sigma)\ \delta[\tau-(\ell + r_T \frac{\dot{\ell}}{c-\dot{\ell}})/c + (1-a)\delta\ \sigma/c]\ \delta(\alpha-1+\dot{\ell}/c)\ d\sigma$$

$$= \frac{c}{(1-a)\delta}\int_{-\infty}^{\infty} f\ \frac{c\sigma}{(1-a)\delta}\ \delta[\tau-\ (\ell + r_T \frac{\dot{\ell}}{c-\dot{\ell}})/c+\sigma]\delta(\alpha-1+\dot{\ell}/c)\ d\sigma$$

$$= \frac{c}{(1-a)\delta}\ f(\frac{-c\tau+\ell+\frac{\dot{\ell}}{c-\dot{\ell}}\ r_T}{(1-a)\delta})\ \delta(\alpha-1+\dot{\ell}/c)\ ,$$

which agrees with (22) since $-(\alpha-1)/\alpha = \dot{\ell}/(c-\dot{\ell})$ from the δ function. For $\beta \neq o$

$$S(\tau,\alpha) = \int_{-\infty}^{\infty} f(\sigma)\ \delta\{\tau-\ell/c+[(1-a)\delta\sigma + \frac{\alpha_p(\sigma)-1}{\alpha_p(\sigma)}\ r_T]/c\}\delta(\alpha-1+\frac{\dot{\ell}}{c}+\frac{\beta\ \delta\sigma}{c})\ d\sigma$$

$$= \frac{c}{\beta\delta}\ f[\frac{c}{\beta\delta}(1-\alpha-\dot{\ell}/c)]\ \delta\{\tau-\frac{\ell}{c} + \frac{1}{\beta}(1-a)(1-\alpha-\frac{\dot{\ell}}{c}) + \frac{\alpha-1}{\alpha}\ \frac{r_T}{c}\}\ .$$

If we solve for $\frac{c}{\beta\delta}\ (1-\alpha-\frac{\dot{\ell}}{c})]$ from the argument of the δ function and substitute the result into the argument of f we get

$$S(\tau,\alpha) = \frac{c}{\beta\delta}\ f(\frac{-c\tau+\ell-\frac{\alpha-1}{\alpha}\ r_T}{(1-\alpha)\delta})\ \delta\{\tau-\frac{\ell}{c} + \frac{1}{\beta}(1-a)(1-\alpha+\frac{\ell}{c})+\frac{\alpha-1}{\alpha}\ \frac{r_T}{c}\}$$

Finally from the property that $|a|\delta(at) = \delta(t)$, (22) results.

APPENDIX G

DERIVATION OF THE MATCHED FILTER OUTPUT FOR THE BISTATIC ECHO FROM A MOVING, TURNING TARGET IN A NON-DISPERSIVE MEDIUM

Substitution of (19) and (22) into the fundamental relationship of Fig. 2 and (5b)

$$\phi_3^D(\tau,1) \approx \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-2\pi j\left(\frac{\alpha'-\alpha_0}{\alpha_0}\right)\frac{f_0^2}{k}} \gamma_{FM}^D[\alpha_0(\tau-\tau'+\frac{\alpha'-\alpha_0}{\alpha_0^2}\frac{f_0}{k}+\frac{\alpha'-1}{\alpha'}t_T)1/\alpha_0]$$

$$\cdot \frac{c}{(1-a)\delta} f\left(\frac{-c\tau'+\ell-\frac{\alpha'-1}{\alpha'}r_T}{(1-a)\delta}\right) \delta[\alpha'-\alpha_0-\frac{\beta}{1-a}(\tau'-\frac{\ell}{c}+\frac{\alpha'-1}{\alpha'}\frac{r_T}{c})]\, d\alpha'\, d\tau'$$

where $\alpha_0 = 1-\dot{\ell}/c$. Interchanging the order of integration, making a dummy variable change from τ' to $\tau'-\frac{\ell}{c}+\frac{\alpha'-1}{\alpha'}\frac{r_T}{c}$, and reversing the order of integration again yields

$$\phi_3^D(\tau,1) \approx \frac{c}{(1-a)\delta}\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-2\pi j\left(\frac{\alpha'-\alpha_0}{\alpha_0}\right)\frac{f_0^2}{k}} \gamma_{FM}^D[\alpha_0(\tau-\tau'-\frac{\ell}{c}+\frac{\alpha'-1}{\alpha'}\Delta t_T$$

$$+\frac{\alpha'-\alpha_0}{\alpha_0^2}\frac{f_0}{k}),\ 1/\alpha_0] \cdot f\left(\frac{-c\tau'}{(1-a)\delta}\right) \delta\left[\alpha'-\alpha_0-\frac{\beta\tau'}{1-\alpha}\right] d\alpha\, d\tau'$$

$$= \frac{c}{(1-a)\delta}\int_{-\infty}^{\infty} e^{-2\pi j\frac{\beta\tau'}{(1-a)\alpha_0}\frac{f_0^2}{k}} f\left(\frac{-c\tau'}{(1-\alpha)\delta}\right)$$

$$\cdot \gamma_{FM}^D\{\alpha_0\left[\tau-\tau'-\frac{\ell}{c}+(1-\frac{1}{\alpha_0+\frac{\beta\tau'}{1-a}})\Delta t_T+\frac{\beta\tau'}{(1-a)\alpha_0^2}\frac{f_0}{k}\right],1/\alpha_0\}\, d\tau' .$$

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But, as pointed out in the main text, Δt_T must be small; therefore the approximation

$$\frac{1}{\alpha_o + \frac{\beta\tau'}{1-a}} \approx \frac{1}{\alpha_o} - \frac{\beta\tau'}{(1-a)\alpha_o^2}$$

may be used. If it is used and the dummy variable τ' replaced by $A\tau'$ where

$$A = 1 - \frac{\beta}{(1-a)\alpha^2_o}\left(\Delta t_T + \frac{f_o}{k}\right) ,$$

the result is

$$\phi_3^D(\tau, 1) \approx \frac{c}{A(1-a)\delta} \int_{-\infty}^{\infty} \gamma_{FM}^D\left[\alpha_o\left(\tau-\tau' - \frac{\ell}{c} + \frac{\alpha-1}{\alpha_o}\Delta t_T\right), 1/\alpha_o\right] ,$$

$$\cdot\ e^{-2\pi j \frac{\beta\tau'}{A(1-a)\alpha_o}\frac{f_o^2}{k}} f\left(\frac{-c\tau'}{A(1-a)\delta}\right) d\tau' .$$

It is convenient rewrite the last result in terms of a delayed convolution

$$\phi_3^D(\tau, 1) \approx \psi(\tau - \tau_d),$$

$$\psi(\tau) = \int_{-\infty}^{\infty} g(\tau - \tau')\, h(\tau')\, d\tau' ,$$

where τ_d is given in Eq. 26 (recall $\alpha_o = 1 - \dot{\ell}/c$) and

$$h(\tau) = \frac{c}{A(1-a)\delta}\, e^{-2\pi j \frac{\beta\tau}{A(1-a)\alpha_o}\frac{f_o^2}{k}} f\left(\frac{-c\tau}{A(1-a)\delta}\right),$$

$$g(\tau) = \gamma_{FM}^{D}\left[\alpha_o\left(\tau - \frac{\alpha_o - 1}{\alpha_o}\,\frac{f_o}{k}\right),\; 1/\alpha_o\right]$$

$$= e^{2\pi j\left(\frac{\alpha_o+1}{2}\right) f_o\left(\tau - \frac{\alpha_o-1}{\alpha_o}\,\frac{f_o}{k}\right)}\,\gamma_{FM}^{M}(\tau,\alpha_o)$$

from Eq. 12. Fourier transforming $\psi(\tau)$ yields

$$(\Psi) = \mathcal{F}[\psi(\tau)]$$

$$= B\,\Gamma_{FM}^{M}(f - f_o', \alpha_o)\,H(f)$$

where $\Gamma_{FM}^{M}(f,\alpha)$ and f_o' are defined in Eq. 26 and

$$H(f) \triangleq \mathcal{F}[h(\tau)]$$

$$B \triangleq e^{-2\pi j\, f_o'\,\frac{\alpha_o-1}{\alpha_o}\,\frac{f_o}{k}} .$$

But if $\Psi^M(f)$ and $H^M(f)$ are defined by

$$\Psi(f) = B\,\Psi^M(f - f_o')$$

$$H(f) = H^M(f - f_o')$$

then clearly $\Psi^M(f)$ obeys the equation given for it in Eq. 26; furthermore inverse Fourier transformation of $\Psi(f)$ yields

$$\psi(\tau) = e^{2\pi j\, f_o'\left(\tau - \frac{\alpha_o-1}{\alpha_o}\,\frac{f_o}{k}\right)}\,\psi^M(\tau)$$

or

$$\phi_3(\tau,1) \quad e^{2\pi j\, f_o'\left(\tau-\tau_d - \frac{\alpha_o-1}{\alpha_o}\,\frac{f_o}{k}\right)}\,\psi^M(\tau - \tau_d),$$

which agrees with Eq. 25 if τ_d' is as given in Eq. 26.

To obtain $H^M(f)$ it is necessary to Fourier transform $h(\tau)$. Note that from Eq. 26

$$\mathcal{F}\left[e^{-2\pi j \frac{\beta}{\alpha_o} \frac{\delta\tau}{c} \frac{f_o^2}{k}} f(-\tau)\right] = F\left(f + \frac{\delta}{c} \frac{\beta}{\alpha_o} \frac{f_o^2}{k}\right);$$

therefore

$$H(f) = F\left[\frac{A(1-a)\delta}{c} f + \frac{\delta}{c} \frac{\beta}{\alpha_o} \frac{f_o^2}{k}\right]$$

$$= F\left(\delta\left\{\left[1 - a - \frac{\beta}{\alpha_o^2} \left(At_T + \frac{f_o}{k}\right)\right] f + \frac{\beta}{\alpha_o} \frac{f_o^2}{k}\right\} / c\right)$$

from the definition of A. Frequency shifting this result gives the equation for $H^M(f)$ given in Eq. 26.